Math 362K (Rusin) — FINAL EXAM ANSWERS — May 8 2014.

Please write your name and EID:

Each problem is worth up to 25 points. Most of the points will be awarded based on your reasoning, so be sure to explain (in words) what you are doing.

1. Poker is a card game played with a standard 52-card deck. (Ask me if you don't know what that set of cards looks like.) In the game of poker, each player is given a randomly-selected set of 5 cards from this deck, called his "hand".

(a) What is the probability that a person's hand contains only red cards?

(b) How many hands are a "flush" (a set of 5 cards all of the same suit)?

Extra credit: what is the probability of being dealt a "full house" — a hand consisting of three cards of one number and two cards of another, e.g.

$$\{3\text{club}, 3\text{heart}, 3\text{diamond}, \text{King club}, \text{King spade}\}$$

ANSWER. (a) $\binom{26}{5} / \binom{52}{5} = 253/9996 = 0.0253\ldots$

(b) Count with a tree, asking first "what suit?" and then "which five cards?" to see that the answer is $4 \times \binom{13}{5} = 5148$.

(EC) As in (b) but ask "what number is there three of?", then "what's the other number", then "which three?", then "which two?" to see there are $13 \times 12 \times \binom{4}{3} \times \binom{4}{2} = 3744$ such hands, out of $\binom{52}{5} = 2598960$ hands altogether, making the probability $6/4165$ (about one in 694). Note that full houses are somewhat more rare than flushes, which is why in a game of poker a full house beats a flush.

2. Do you remember Joey with the messy sock drawer? It contains 2 white socks, 2 black socks, 2 red socks, and 16 other socks, none of which match. He needs a pair of matching socks. But now his strategy is: he will draw socks one at a time at random until he has a matching pair (and then he stops).

(a) What is the probability that he succeeds with the very first pair he draws?

(b) What is the probability that he succeeds precisely upon drawing his third sock?

ANSWER. I would view the sample space as being the set of sequences of (different) socks (whose first match includes the last element of the sequence). Note that not all sequences are equally likely, e.g. but all initial ordered pairs are.

(a) Out of the 22×21 sequences of two socks he might have drawn, a total of 6 sequences are a win (six possible socks to be drawn first, and for each there is a unique sock to be drawn second). So the answer is $6/(21 \times 22) = 1/77$.

(b)

You can set this up as a probability tree: there is a 6/22 chance of choosing a "good" sock the first time, and a 16/22 chance of choosing an unmatched sock the first time. In the latter case you have (as in part(a)) a probability of $6/(21 \cdot 20)$ chance of making a match with the next two socks. The other possibility is trickier: If you have first selected a sock from a pair, then the events that constitute a win on the third sock are of different types: you can either select a (different) matching pair on draws 2 and 3 (probability $4/(21 \cdot 20)$), or else the mate of your first sock much show up on the third sock, with any of the other 20 socks appearing as the second sock (probability $(20/21) \cdot (1/20) = (1/21)$). So the total probability is

$$\frac{16}{22}\frac{6}{420} + \frac{6}{22}(\frac{4}{420} + \frac{1}{21}) = 2/77$$

3. In my experience, most UT students are good, hard-working students — all except the 25% of students who just guess on exams (instead of studying), and the 5% of students who cheat. The good students do well: 60% of them turn in above-average exam papers; only 10% of the guessers do so well, as well as 90% of the cheaters.

Sasha just turned in an above-average exam paper. What's the probability that Sasha is a cheater?

ANSWER. Bayes's Theorem. We can compute $Pr(\text{Sasha cheats} \cap \text{Sasha does well}) = .05 \times .9 = 0.045$ but also $Pr(\text{Sasha does well})$ as a sum of the conditional probabilities $\sum_i Pr(\text{Sasha does well} \,|\, \text{Sasha is in group } i) \cdot Pr(\text{Sasha is in group } i) = (.6 \times .70) + (.1 \times .25) + (.9 \times .05) = .49$ Then the conditional probability $Pr(\text{Sasha cheats} \,|\, \text{Sasha does well})$ is the quotient of these: $0.045/0.49 = 9/98 = 0.0918367\ldots$

Informally: Bayes' theorem allows us to see how additional information (Sasha did well!) changes our estimates of probabilities (Sasha's chance of being a cheater nearly doubled once we saw his test result).

4. I don't play darts very well: when I play, the darts go all over the board, hitting the board (a disk with a 20cm radius) with a uniform distribution. Let R be the random variable that measures the distance (in cm) from the dart to the center of the board.

(a) Compute the cumulative distribution function for this random variable.

(b) Compute the pdf for R.

(c) Compute $E[R]$ and $\text{Var}(R)$.

ANSWER. (a) The cdf $F(t)$ measures $Pr(R \leq t) = Pr(\text{dart lies in disk of radius } t) = Area(\text{disk})/Area(\text{board}) = (\pi t^2)/(\pi(20)^2) = (t/20)^2$, for $0 < t < 20$. (Obviously $F(t) = 0$ for $t < 0$ and $F(t) = 1$ for $t > 20$.)

(b) The pdf $f(t)$ is $F'(t) = t/200$ on [0,20] (and is zero outside this interval.)

(c) $E[R] = \int_0^{20} tf(t)\,dt = 20^3/600 = 40/3 = 13.33cm$. Similarly $E[R^2] = \int_0^{20} t^2 f(t)\,dt = 20^4/800 = 200cm^2$ so Var(R)= $E[R^2] - E[R]^2 = 200 - (40/3)^2 = 200/9cm^2$; the standard deviation is 4.714cm.

5. Many clocks being shipped from the factory are tested to find out how long they last before they stop working, so now we know that this random variable has an exponential distribution with parameter $\lambda = \frac{1}{8}$. I just bought a used clock (of unknown age!) from Goodwill. What is the probability it will still be working 8 years after I buy it?

ANSWER. This is a feature of an exponential distribution: that the probability is the same no matter how old the clock when I bought it. So we might as well assume it's brand new, so that we are simply trying to compute $Pr(\text{lifetime} > 8) = \int_8^{\infty} \lambda \cdot e^{-\lambda t}\,dt = e^{-8\lambda} = e^{-1}$, about 37%.

6. A large group of friends splits into sets of six to form Dining Clubs: every week they go out to dinner together and when the bill comes, they roll a die to determine who will pay the (entire) bill — Pat pays for everyone if they roll a "1", Chris pays if they roll a "2", etc. We can define random variables X_1 and X_2 on this large group of friends: X_1 counts the number of meals each person eats for free before the first time they have to pay the bill and similarly X_2 counts the number of free meals they get between the first and second time they pay the bill.

(a) The unluckiest person is the one for whom $X_1 = X_2 = 0$; what is the probability that this event occurs?

(b) Find the joint mass function of X_1 and X_2.

ANSWER. The events X_1 and X_2 are independent (unless you believe that certain people are "(un)lucky"!) so the joint mass function is

$$f(m, n) = Pr(X_1 = m \quad \text{and} \quad X_2 = n) = Pr(X_1 = m)Pr(X_2 = n)$$

But the event $X_1 = m$ happens iff they roll m numbers different from yours and then roll your number: $Pr(X_1 = m) = (5/6)^m(1/6)$. The other factor is similar, so

$$f(m, n) = 5^{n+m}/6^{n+m+2}$$

In particular, $f(0, 0) = 1/36$, as you might expect.

7. The joint density function of two random variables X and Y is given by

$$f(x, y) = \begin{cases} cxy & \text{if } x > 0,\ y > 0, \text{ and } x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute (a) the value of c; (b) the pdf f_Y of the random variable Y;

ANSWER. (a) We need the integral of f on the triangle to be 1. That's

$$1 = \int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx = c\int_0^1 x(1-x)^2/2\,dx = (c/2)\int_0^1 (x-2x^2+x^3)\,dx = (c/2)(1/2-2/3+1/4) =$$

so $c = 24$.

(b) $f_Y(y) = \int_0^{1-y} f(x,y)\,dx = 12y(1-y)^2$.

8. I have just produced Dave's Big Book Of Mazes. Hundreds of mazes for kids to do. My publishers asked how long it takes to complete each one. I pointed out that time-to-completion is actually a random variable X and asked if they wanted the mean μ_X and standard deviation σ_X of this random variable. They said yes. I said I had no idea what the distribution of X was like. They got mad.

So I assembled thousands of volunteers and asked them each to complete 100 of the mazes and to report to me when they were finished. After 30 minutes, 10% of them were done. After another 10 minutes, an additional 20% of them were done. Then I halted the experiment because I now knew μ_X and σ_X! How did I do it?

(Hints: First of all, how does the Central Limit Theorem help here? State any assumptions you make about X.

Secondly, what information do you gain about the expected completion time for 100 mazes when I report what I said about the first 30 minutes? Do you think μ_X is more or less than 0.30 minutes? By a lot or by a little? You should be able to write a useful equation that has both "30min" and μ_X in it, among other things.

Finally, get a similar equation from the information about the next 10 minutes, and then solve these two equations.)

ANSWER. First, CLT tells us that the random variable $Y = X_1+X_2+\ldots+X_{100}$ will be (approximately) normally distributed, with mean equal to $100\mu_X$ and with variance equal to $100\sigma_X^2$ (i.e. $\sigma_Y = 10\sigma_X$). Of course we must assume the hypotheses of the Central Limit Theorem, including the premise that the amount of time needed to complete any one maze is independent of the times needed to complete any others. Moreover, we do have to assume that the initial distribution is "nice" enough that the number of iterations (100) may be considered "large", so that the normal distribution is indeed "close" to the true distribution of Y.

So assume Y is normally distributed with these parameters. Then probabilities like $Pr(Y < t)$ should be tabulated by the cdf for the normal distribution; in particular, we are given that $Pr(Y < 30min) = 0.10 = Pr(Z < -1.28)$ (using a calculator or the tables for the normal distribution), i.e. this value of 30 min is apparently 1.28 standard deviations below the mean for Y, i.e.

$$30min = 100\mu_X - 12.8\sigma_X$$

In the same way the other datum shows us that

$$40min = 100\mu_X - 5.3\sigma_X$$

so we can subtract and conclude $10min = 7.5\sigma_X$, i.e. $\sigma_X = 4/3min$ and then $30min = 100\mu_X - (12.8)(1.33min)$ makes $\mu_X = 0.85min$.

In truth, these data are a little suspicious: since X is clearly a non-negative random variable, we don't expect the standard deviation (1.33min) to be so large compared to the mean (0.85min) unless the distribution has a fairly long tail. Evidently it happens with some regularity that there are some very difficult mazes! Another way to say this is that after averaging 100 independent values of a random variable, we expect to see the reported values of the averages very close to the true mean value; that means that once a few people have started to report finishing their 100 mazes, we should see all the others cascading in, in very short order.

(An example of a distribution with roughly these characteristics is $f(x) = 1.615 - 1.567x$ on [0,1] and then $f(x) = .048 - .007(x - 1)$ on [1,8].)