M365C (Rusin) HW10 – some comments

1. The function is integrable. I can show this by finding a partition P for which the upper sum $U(P, f)$ is less than any pre-assigned ϵ.

Note first that the set

$$S = \{r \in \mathbf{Q} \mid f(r) > \epsilon/2\}$$

has some finite cardinality N. (We would need $r = p/q$ with $0 \le p \le q$ and $q < 2/\epsilon$ and there are only finitely many such pairs (q, p).) So pick a $\delta > 0$ which is less than $\epsilon/(2N)$ and is also less than the minimum of all the distances $|r - r'|$ for distinct points $r, r' \in S$. Then each of the intervals $[r, r+\delta]$ contains just one element of S, namely its left endpoint.

So now I can show $U(P, f) = \sum M_i(x_i - x_{i-1})$ will be less than ϵ, where P is the partition containing all the points $r \pm \delta/2$ for $r \in S$ On each one of the little intervals $[r-\delta/2, r+\delta/2]$ the function of course never takes a value greater than 1, so the summands for these intervals are all at most δ, and hence the sum of all these summands will be at most $N\delta < \epsilon/2$. Then on the gaps between these little intervals, there are no elements of S at all, so the supremum of f there is at most $\epsilon/2$ (from the definition of S); the sum of all *these* summands is then at most $\epsilon/2$ times the sum of their lengths (which must be less than the length of $[0, 1]$), and so this sum is also at most $\epsilon/2$. Putting all the summands together shows $U(P, f) < \epsilon$, as desired.

You might notice the similarity to the proof that $h \circ f$ is integrable if f is (for continuous h): there's a lot of tall "bad" intervals but they are skinny, and then there are lots of "good" intervals but they are all short, so in toto, $U(P, f)$ is small.

It is a drawback to Riemann's definition of the integral that we are forced to construct the little intervals around the rational numbers in such a way that the intervals stay disjoint (so that we may track the small intervals in the resulting partition). In spirit one should expect something like this: if f and g are two functions that agree everywhere except at some countable collection of points x_n, then the integrals of f and g should be equal (including the statement that each is integrable iff the other is). The rationale behind this thinking is that we can find an interval of width $\epsilon/2^n$ around x_n, and then the sum of the lengths of these intervals is a geometric series summing to ϵ; in other words the total length of the union of these intervals can be made as small as we like, and so in the Riemann sums the contributions of the regions where f and g differ can be made negligible. Sadly, you won't be able to get all the details to quite work out: we already know that the function which is equal to 1 on $\mathbf{Q}$ and 0 otherwise is not even integrable, even though this differs from the (integrable!) zero function only at countably many points. Nonetheless this idea of wrapping up a countable collection of "bad" points in a negligible set is powerful, and leads to the alternative Lebesgue defintion of the integral, which we will leave outside the syllabus for this course!

2. Simply take $f(x) = 1$ for rational x and $f(x) = -1$ for irrational x; then f^2 is integrable over any finite interval but f is not.

3. Since f is everywhere-positive, $F(T) = \int_0^T f(t)\,dt$ is an increasing function, and the partial sums $S_N = \sum_{n\geq 0}^N f(n)$ is an increasing sequence. Therefore each of these has a limit iff it stays bounded.

So in the one direction, if we know the series converges to some value L, then each S_N is at most L. On the other hand since f is decreasing, for each integer $n \in N$ we have $f(n) \geq \int_{[n,n+1]} f$, and so if we add these inequalities for $n = 0, 1, \ldots, N-1$ then we get $S_{N-1} \geq \int_{[0,N]} f$. So all for any real T, let $N = \lceil T \rceil$ and then $F(T) \leq F(N) \leq S_{N-1} \leq L$. So the values of F are bounded by L, so their limit exists as $T \to \infty$.

The proof in the opposite direction is similar: if each integral is less than L then so is each $S_N - f(0) = \sum_{n\geq 1}^N f(n)$ since $f(n) \leq \int_{n-1}^n f$. Then all the partial sums S_N are bounded by $L + f(0)$, so the incresing sequence of them converges.

4. You may compose the integrable function f with the continuous function $h(x) = |x|$ and use the theorem proved in class!

5. If there were a point c in the interval where the function took on a value $M > 0$ then simply use the definition of continuity at c to find an interval $(c - \delta, c + \delta)$ where f differs from M by at most $\epsilon = M/2$. Then $\int_J f \geq (2\delta)(M/2) > 0$, contradicting the premise.

I concede my language was a little sloppy: a single point c is actually an interval $J = [c, c]$, and no matter what the function f is, $\int_J f = 0$ (whether or not $f = 0$).