M365C (Rusin) HW10 – due Thursday, Nov 21 2019

1. Let

$$f(x) = \begin{cases} 1/q & \text{if } x = p/q \in \mathbf{Q}, \text{ in lowest terms} \\ 0 & \text{if } x \in \mathbf{R} - \mathbf{Q} \end{cases}$$

Determine the value of the integral $\int_{[0,1]} f$, or prove that the function is not integrable.

2. If f^2 is integrable on an interval, must f also be integrable? (You cannot use theorem 6.11 because we don't know that $f = \sqrt{f^2}$, because we don't know f is everywhere-positive.)

3. Prove the Integral Test from Calculus: given a function $f : [0, \infty) \to \mathbf{R}$ which is (i) everywhere-positive and (ii) everywhere-decreasing, then the infinite series $\sum_{n \geq 0} f(n)$ converges iff the limit

$$\lim_{T \to \infty} \int_0^T f(t)\, dt$$

exists.

4. Show that if f is integrable on an interval J then so is $|f|$, and

$$\left| \int_J f \right| \leq \int_J |f|$$

5. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous. Show that if $f(x) \geq 0$ on an interval J and $\int_J f = 0$, then $f(x) = 0$ for all $x \in J$. Does the conclusion hold if we weaken the hypothesis from "continuous" to "integrable"?