M365C (Rusin) Exam 2 — some comments.

1. Suppose $f : \mathbf{R} \to \mathbf{R}$ is increasing on the intervals (a, c) and (b, d), where $a < b < c < d$. Show that f is increasing on (a, d).

ANSWER: I must show that if $x, y \in (a, d)$ and $x < y$ then $f(x) < f(y)$. First pick a point in the overlap between the intervals, e.g. the point $e = (b + c)/2$.

1. If $y \leq e$ then both x and y lie in (a, c) so by assumption $f(x) < f(y)$.
2. If $x \geq e$ then similarly $f(x) < f(y)$ because f is increasing on (b, d).
3. If neither of those is true, then $x < e < y$. But in that case we use the fact that f is increasing on the first interval to show $f(x) < f(e)$ and then use the fact that f is increasing on the second interval to show that $f(e) < f(y)$. Taken together these prove $f(x) < f(y)$.

2. A function $f : \mathbf{R} \to \mathbf{R}$ is said to satisfy the *Lipschitz condition* if there is a constant C such that for every x and y in $\mathbf{R}$, $|f(x) - f(y)| \leq C|x - y|$. Show that a function that satisfies the Lipschitz condition is uniformly continuous on its domain.

ANSWER: Given an $\epsilon > 0$ let $\delta = \epsilon/C$. Then whenever $x, y \in \mathbf{R}$ satisfy $|x - y| < \delta$, we will have $|f(x) - f(y)| \leq C|x - y| < C\delta = \epsilon$.

This argument does not apply if $C = 0$, but if $C = 0$ then the Lipschitz condition implies that f is a constant function, which is clearly uniformly continuous. (It is perhaps also worth noting that *no* function can satisfy a Lipschitz condition with $C < 0$ because absolute values are always non-negative.)

Extra Credit: Suppose instead that there is a constant C such that for every x and y in $\mathbf{R}$ we have $|f(x) - f(y)| \leq C|x - y|^2$. Show that f is constant.

ANSWER: Given any two points $x, y \in \mathbf{R}$ with, say, $x < y$, consider the $n + 1$ equally-spaced points $x_i = x + i(y - x)/n$ (so $x_0 = x$ and $x_n = y$. By assumption we have $|f(x_i) - f(x_{i-1})| \leq C|x_i - x_{i-1}|^2 = C(y - x)^2/n^2$ for every i. Adding these and using the triangle inequality shows $|f(y) - f(x)| < n \cdot C(y - x)^2/n^2 = C(y - x)^2 \cdot (1/n)$ This is true for every $n \in \mathbf{N}$ so the actual value of $|f(y) - f(x)|$ must be no larger than the infimum of all these numbers, which is zero, so $f(x) = f(y)$. Hence f is constant.

Alternatively you could use the given property to show that for every $x \in \mathbf{R}$, the derivative $f'(x)$ exists and equals zero. But the Mean Value Theorem implies that a function whose derivative is everywhere zero must be constant.

3. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a continuous function with the property that for all $x, y \in \mathbf{R}$ we have $f(x - y) = f(x) - f(y)$. Suppose also that $f(1) = 7$. Show that $f(x) = 7x$ for every x. (Hint: Choose combinations of x and y that allow you to determine these values: $f(0)$, $f(-1)$, $f(2)$, $f(694)$, $f(\frac{1}{2})$, $f(3.14)$, and $f(\pi)$. Generalize what you learn from these examples.)

ANSWER: With $x = y = 1$ we conclude $f(0) = 0$. Then with $x = 0$ we deduce $f(-y) = -f(y)$ for every real y. In particular $f(x + y) = f(x - (-y)) = f(x) - f(-y) = f(x) + f(y)$ for every x and y. (One says such an f is an *additive function*.) By induction we prove $f(nx) = nf(x)$ for every real x and every $n \in \mathbf{N}$: it's already proved when $n = 0$ and the inductive step is simply

$$f((n+1)x) = f(nx + x) = f(nx) + f(x) = nf(x) + f(x) = (n+1)f(x)$$

In particular taking $x = 1$ shows $f(m) = 7m$ for every $m \in \mathbf{N}$. We can also take $x = y/n$ for any real y to conclude $f(y) = nf(y/n)$ so that $f(\frac{1}{n}y) = \frac{1}{n}f(y)$. So if $y = m \in \mathbf{N}$ we have $f(m/n) = f(m)/n = 7m/n$. Thus $f(q) = 7q$ for every positive rational q, as well as for $q = 0$ and for negative rationals (since $f(-x) = -f(x)$).

Finally (*finally!*) we use continuity: every $x \in \mathbf{R}$ is a limit of rational numbers x_n, so by continuity $f(x) = f(\lim x_n) = \lim f(x_n) = \lim 7x_n = 7x$. Thus $f(x) = 7x$ for all $x \in \mathbf{R}$. (You may wish to remember this principle: if f and g are two continuous functions that agree on all of $\mathbf{Q}$ then they are equal everywhere. Simply note $h = f - g$ is also continuous and the closed set $h^{-1}(0)$ includes all of $\mathbf{Q}$ so it includes $\bar{\mathbf{Q}} = \mathbf{R}$.)

In this problem you MUST use continuity; you MUST NOT use differentiability (since we are not told f is differentiable). To see that continuity is required, view $\mathbf{R}$ as an (infinite-dimensional) vector space over $\mathbf{Q}$ and pick a basis for $\mathbf{R}$ which includes 1. Then we may define a $\mathbf{Q}$-linear map $\mathbf{R} \to \mathbf{R}$ by simply stating where the basis elements are to be sent. As long as 1 is sent to 7, then the conditions of the problem are met (other than continuity); we could for example send all the other basis elements to 0 so that f is a projection from $\mathbf{R}$ to $\mathbf{Q}$ which sends every rational x to $7x$ but sends most other real numbers to 0!

4. Suppose that $f, g : \mathbf{R} \to \mathbf{R}$ are two differentiable functions with $f'(x) > g'(x)$ for all $x > 0$. Show that if $f(0) = g(0)$ then $f(x) > g(x)$ for all $x > 0$.

ANSWER: Let $h(x) = f(x) - g(x)$; then h is differentiable and $h'(x) > 0$ for all x. We have previously shown that this means h is increasing (by the Mean Value Theorem), which means $h(x) > h(0) = 0$ for all $x > 0$. Hence $f(x) > g(x)$ for all these x.

Note that if you apply MVT to f and g separately, you can only assert that there are *two* points c_1 and c_2 where the derivatives of f and g match their secant slopes; there's no reason to think $c_1 = c_2$, so you can't use the fact that $f' > g'$ pointwise. Also note that there is no assumption that f' and g' are positive; for example the claim applies to $f(x) = x^2 \sin(1/x)$ and $g(x) = f(x) - f(x)^2$. In particular f and g need not be increasing nor decreasing. And no, the generalized MVT found in the text is not useful in this problem.

5. Suppose $f : \mathbf{R} \to \mathbf{R}$ is twice differentiable, and suppose that at the point $a \in \mathbf{R}$ we have $f'(a) = 0$ and $f''(a) > 0$. Show that f attains a local minimum at the point a, that is, show that there is a neighborhood N around a with the feature that for every $x \in N$ we have $f(x) \geq f(a)$. You may assume that the second derivative of f is continuous.

ANSWER: It's not quite enough to know that $f''(a) > 0$; it's better to know that f'' is positive in a neighborhood of a, but that's a consequence of the continuity of f'', as we have seen in multiple other problems. (To be precise: use the definition of continuity of f'' at the point a, with $\epsilon = f''(a)$; then there is a δ such for all x in $(a - \delta, a + \delta)$ we have $f''(x) > f''(a) - \epsilon = 0$.)

Then, for all x in this neighborhood, we may invoke the Remainder Theorem for Taylor Series:

$$f(x) = f(a) + f'(a)\,(x-a) + \frac{f''(c)}{2}\,(x-a)^2$$

for some c between a and x. That means c will also be in this neighborhood, so that $f''(c) > 0$ as in the previous paragraph. Since $(x-a)^2$ is obviously also positive (for $x \neq a$) and $f'(a) = 0$ by assumption, this leaves us with $f(x) \geq f(a)$.

It turns out the result is true without the assumption that f'' be continuous. I invite you to consider whether the function

$$f(x) = \begin{cases} 3x^4 \sin(1/x) \; + x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

has a local minimum at the origin. You can show that $f''(0) = 2$ from the definition; and separately you can compute that $f''(x) = (36x^2 - 3)\sin(1/x) - 18x\cos(1/x) + 2$ if $x \neq 0$. Put together, these computations show that this f'' is not continuous at 0. Well? Is there a region around 0 where the function stays positive? Do observe that if $x = 1/(\pi/2 + 2n\pi)$ for $n = 1, 2, 3, \ldots$, then $f''(x) = 36x^2 - 1 < 0$, that is, there is no neighborhood around 0 where f'' stays positive!

So instead we may proceed as follows. The definition of $f''(a)$ is $\lim_{x\to a}(f'(x) - f'(a))/(x-a)$. Apply the definition of a limit, with $\epsilon = f''(a)/2$; then there is a neighborhood $N = (a-\delta, a+\delta)$ such that for all x in this neighborhood we have $(f'(x) - f'(a))/(x-a) > f''(a) - \epsilon > f(a)/2 > 0$. Then in particular for all $x > a$ in this neighborhood, $f'(x) - f'(a) = f'(x)$ must be positive; similarly for all $x < a$ in N, $f'(x) - f'(a) = f'(x)$ must be negative (because the denominator $x - a$ is). Then for any $x \in N$ we may apply the Mean Value Theorem: $f(x) - f(a) = f'(c)(x-a)$ for some c between a and x; for $x > a$ both $f'(c)$ and $x - a$ are positive and for $x < a$ both are negative, but either way the product is positive so $f(x) > f(a)$. So indeed f attains a local minimum at a.

It's fine to speak of a function (like this f') being increasing in a neighborhood of a point, if you can identify this neighborhood somehow. If you can't, and you speak only of a function "increasing at the point a" or something, then I don't know what you're talking about and I'm not sure you do either.