Math 408D (Rusin) — FINAL EXAM — May 17 2011. Here are some answers.

1. Calculate $\lim_{x\to\infty} 4x(e^{1/x} - 1)$

Answer: This is an indeterminate form of the "$\infty \cdot 0$" type. Write the function instead in the form $\dfrac{4(e^{1/x}-1)}{1/x}$ and apply L'Hopital's Rule: the limit is the same as that of

$$\frac{4(e^{1/x})(-1/x^2)}{-1/x^2} = 4e^{1/x} \longrightarrow 4$$

2. For what positive numbers a does this improper integral converge? Explain.

$$\int_a^\infty \frac{4}{(x-3)^2}\,dx$$

Answer: The integral is (potentially) improper in both senses: on (say) the interval $[10,\infty)$ the function is continuous but the region of integration is unbounded so the integral over this interval is defined to be

$$\lim_{T\to\infty} \int_{10}^T \frac{4}{(x-3)^2}\,dx$$

which can be easily evaluated by a u-substitution: if $u = x-3$, the antiderivative is $-4/u$ and so the integral is $-(4/(T-3)) + (4/7)$, which approaches $4/7$ as $T\to\infty$.

Separately, on the interval $[a,10]$ we have a bounded interval but an integrand $f(x)$ which is not continuous on $[a,10]$ if $a \le 3$. So the integral over this interval is, for such a, defined as

$$\lim_{L\to 3^-} \int_a^L f(x)\,dx + \lim_{R\to 3^+} \int_R^{10} f(x)\,dx$$

assuming both those limits exist. As above, the antiderivative is $-4/(x-3)$ so we need to assess the limits

$$\lim_{L\to 3^-} \frac{-4}{L-3} - \frac{-4}{a-3} \qquad \text{and} \qquad \lim_{R\to 3^+} \frac{-4}{10-3} - \frac{-4}{R-3}$$

But neither of these limits exists (they both diverge to $+\infty$) so the integral does not converge for any $a \le 3$.

3. Let $\{a_n\}$ be the sequence whose nth term is $a_n = \sqrt{(n^2+6n)} - n$. Determine whether this sequence converges, and if so, to what.

Answer: You can turn this into a L'Hopital's Rule problem of the "$\infty - \infty$" type and then into the "$\infty \cdot 0$" type, but I think it's faster to rewrite a_n as

$$\frac{(n^2+6n)-(n)^2}{\sqrt{(n^2+6n)}+n}$$

and then view this as a "∞/∞" kind of problem. Applying L'Hopital's Rule we get the same limit as that of $6/(\frac{n+3}{\sqrt{n^2+6n}}+1)$. That fraction in the denominator is the square root of $(n^2+6n+9)/(n^2+6n)$ and hence approaches 1, which means the denominator approaches 2 while the numerator is just 6, making the fraction tend towards 3. By L'Hopital's Rule, our original limit is then also equal to 3.

4. Evaluate $\sum_{n=0}^{\infty} \frac{1+3^n}{7^n}$

Answer: This is the sum of two geometric series,

$$\sum_{n=0}^{\infty} \frac{1}{7^n} + \sum_{n=0}^{\infty} \left(\frac{3}{7}\right)^n$$

and so the sum is $\frac{1}{1-(1/7)} + \frac{1}{1-(3/7)} = \frac{35}{12}$.

5. Is this series convergent or divergent? $\sum_{m\geq 0} \frac{1+(-1)^n}{n!}$

Answer: It's convergent. You could do a comparison with $\sum_{m\geq 0} \frac{1+(+1)^n}{n!} = \sum_{m\geq 0} \frac{2}{n!}$; you can even recognize the series as the sum of two parts, $\sum_{m\geq 0} \frac{1}{n!} + \sum_{m\geq 0} \frac{(-1)^n}{n!} = e + e^{-1}$.

6. Find the interval of convergence for the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}(x+1)^k$

Answer: If you try, say, the Ratio Test, you find that the series converges absolutely if $\lim_{k\to\infty} \frac{(1/(k+1)^2)|x+1|^{k+1}}{(1/k^2)|x+1|^k} < 1$ and diverges if this limit is greater than 1. Since $\lim \frac{k}{k+1} = 1$, the limit is less than 1 iff $|x+1| < 1$, i.e. iff $x \in (-2, 0)$. If $x < -2$ or $x > 0$ then the limit is more than 1, and the series diverges.

If $x = -2$, the series is simply $\sum_{k=1}^{\infty} \frac{1}{k^2}$, which is convergent by the integral test. (It's a "p-series", with $p > 1$.) If $x = 0$, the series is $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$, which is strictly alternating and decreasing, hence again convergent by the Alternating Series Test.

So the interval of convergence is exactly $[-2, 0]$.

7. Give a power series representation of $f(x) = \frac{2x}{1-x^2}$.

Answer: This $f(x)$ matches the presentation of the sum of a geometric series starting with $2x$ and having common ratio x^2, so $f(x) = 2x + 2x^3 + 2x^5 + \ldots$

You could use Partial Fractions and write this function as $\frac{1}{1-x} - \frac{1}{1+x}$, which we similarly recognize as geometric series: $\sum_{n\geq 0} x^n - \sum_{n\geq 0}(-x)^n = \sum_{n\geq 0}(1-(-1)^n)x^n = \sum_{n\geq 0} 2x^{2k+1} = 2x + 2x^3 + 2x^5 + \ldots$.

Or you might have recognized f as the derivative of $-\ln(1-x^2)$, so you could write the series for the latter, and then differentiate.

Just don't attempt to use Taylor's Formula ...

8. Find the degree-3 Taylor polynomial T_3 centered at $x = 25$ for the square-root function $f(x) = \sqrt{x}$. Use your polynomial to estimate $\sqrt{26}$.

Answer: The derivatives of $f(x) = x^{1/2}$ are $f'(x) = (1/2)x^{-1/2}, f''(x) = (-1/4)x^{-3/2}$, $f'''(x) = (3/8)x^{-5/2}$, etc. The values of these functions at 25 are, respectively, 5, 1/10, $-1/500$, and 3/25000. So the Taylor polynomial we want is

$$T_3(x) = 5 + (1/10)(x-25) + (-1/1000)(x-25)^2 + (1/50000)(x-25)^3.$$

When $x = 26$ this gives us the estimate

$$f(26) \approx T_3(26) = 5 + (1/10) + (-1/1000) + (1/50000) = 5 + .1 - .001 + .00002 = 5.09902.$$

The correct value is $\sqrt{26} = 5.0990195135927848300282241090 2\ldots$ Approximately :-) .

9. The equation (in polar coordinates) $r = \cos^2(\theta)$ defines a curve in the xy-plane. Find the equation of the line tangent to the curve at the point on that curve having $\theta = \pi/4$.

Answer: The point has $r = \cos^2(\pi/4) = 1/2$, and so its Cartesian coordinates are $x = r\cos(\theta) = \sqrt{2}/4$ and $y = \sqrt{2}/4$. Now we just need the slope of the line, which is (using the Chain Rule to compute dx and dy, and then noting that $dr = -2\cos(\theta)\sin(\theta)d\theta$

$$\frac{dy}{dx} = \frac{\sin(\theta)dr + r\cos(\theta)d\theta}{\cos(\theta)dr - r\sin(\theta)d\theta} = \frac{-\sin(\theta)2\cos(\theta)\sin(\theta) + r\cos(\theta)}{-\cos(\theta)2\cos(\theta)\sin(\theta) - r\sin(\theta)}$$

When $\theta = \pi/4$ and $r = 1/2$ this comes out to 1/3, so the line is $(y - \sqrt{2}/4) = \frac{1}{3}(x - \sqrt{2}/4)$.

10. Consider the triangle whose vertices are the points $P = (2,1,0)$, $Q = (3,-1,2)$, and $R = (4,-1,-1)$. Find (a) the area of this triangle, and (b) the angle in this triangle at the vertex P.

Answer: Two sides of the triangle form the vectors $PQ = (1,-2,2)$ and $PR = (2,-2,-1)$ respectively. The area of the triangle is half that of the corresponding parallelogram, i.e.

$(1/2)\,||PQ \times PR|| = (1/2)||6i + 5k + 2k|| = \sqrt{65}/2$. The angle at P has a cosine equal to $(PQ \cdot PR)/(||PQ|| \cdot ||PR||) = 4/(3 \cdot 3)$, that is, the angle is $\arccos(4/9)$.

11. Find the point on the plane $x + 2y - z = 2$ that is closest to the origin.

Answer: The distance from a point (x, y, z) to the origin is $\sqrt{x^2 + y^2 + z^2}$; since on this plane we have $z = x+2y-2$, our job is to find a pair (x, y) which minimizes the expression $\sqrt{x^2 + y^2 + (x + 2y - 2)^2}$. It's actually sufficient to minimize the square of this expression, $x^2 + y^2 + (x + 2y - 2)^2 = 2x^2 + 4xy + 5y^2 - 4x - 8y + 4$. The minimum occurs when the gradient is zero, i.e. when $4x + 4y - 4 = 4x + 10y - 8 = 0$. That gives two linear equations in two unknowns, whose unique solution is $x = 1/3, y = 2/3$ (where $z = -1/3$). So there is only this one critical point, and it is indeed the location of a (local, hence global) minimum: the Hessian matrix is

$$\begin{pmatrix} 4 & 4 \\ 4 & 10 \end{pmatrix}$$

with determinant $24 > 0$ and trace $14 > 0$. So the point $(x, y, z) = (1/3, 2/3, -1/3)$ is on the plane and closer to the origin than any other.

12. A certain curve is parameterized by $x = 2t^2, y = \cos(\pi t), z = e^{t-1}$. There is a plane which passes through the origin and also is tangent to this curve at the point $(2, -1, 1)$. Find the equation of this plane.

Answer: We pass through that point P only when $t = 1$. At that moment, the tangent line points in the direction of the velocity vector $(x', y', z') = (4t, -\pi \sin(\pi t), e^{t-1}) = (4, 0, 1)$. So the normal vector is perpendicular to both this vector and to the vector $OP = (2, -1, 1)$, which means the normal vector is parallel to their cross product, $(1, -2, -4)$. Therefore the plane is of the form $x - 2y - 4z = d$ for some number d but obviously $d = 0$ since the plane passes through the origin.

13. Suppose $w = f(x, y, z)$ where f is a function with partial derivatives given by $f_x = ye^x + yz^3, f_y = e^x + xz^3 + z$, and $f_z = 3xyz^2 + y$. Also suppose that x, y, and z are themselves functions of time t, so that w also varies with time. If $x(0) = 0, y(0) = 1, z(0) = 2, x'(0) = 3, y'(0) = 4$, and $z'(0) = 5$, what is $w'(0)$?

Answer: $w'(0) = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = 9 \cdot 3 + 3 \cdot 4 + 1 \cdot 5 = 44$

Incidentally, the function f has to differ by a constant from $ye^x + xyz^3 + yz$.

14. The surfaces $xy^3z^4 = 1$ and $xyz = 1$ intersect at the point $(-1, -1, 1)$. Find the angle between the surfaces at this point. (Hint: that means the angle between the two tangent planes there, which equals the angle between the two planes' normal lines.)

Answer: The gradient to the surface $xyz = 1$ is (yz, xz, xy), which means in particular that at the point $(-1, -1, 1)$ the vector $(-1, -1, 1)$ points perpendicular to the surface. Likewise the gradient to the other surface at (x, y, z) is $(y^3z^4, 3xy^2z^4, 4xy^3z^3)$ and at our point that vector is $(-1,-3,4)$. The angle between the surfaces is the angle between these vectors, an angle with cosine $8/(\sqrt{3}\sqrt{26})$, so that our angle is $\arccos(8/\sqrt{78})$.

15. Minimize $x^2+2y^2+2z^2$ subject to the constraint $x+y+z=1$.

Answer: You can do this by eliminating z (say) or use Lagrange Multipliers: the minimum value occurs when $\nabla f=(2x,4y,4z)$ is parallel to $\nabla g=(1,1,1)$. That requires $2x=4y=4z$ (as well as $x+y+z=1$) so $(x,y,z)=(1/2,1/4,1/4)$.

In the remaining three problems, evaluate the integral $\iint_A f(x,y)\,dxdy$:

16. $f(x,y)=x^2$ and $A=\{(x,y):1\le x\le 2, 3\le y\le 5\}$.

Answer: It will be $(5-3)\int_1^2 x^2\,dx=14/3$.

17. $f(x,y)=x^2+y$ and $A=\{(x,y):0\le x\le 2, -x\le y\le x\}$.

Answer: This is

$$\int_0^2\int_{-x}^{x}(x^2+y)dydx=\int_0^2(x^2y+\frac{y^2}{2})|_{y=-x}^{x}\,dx=\int_0^2 2x^3dx=8$$

18. $f(x,y)=x^2+y^2$ and $A=\{(x,y):0\le x\le 2, -\sqrt{4-x^2}\le y\le\sqrt{4-x^2}\}$.

Answer: Use polar coordinates: the integral equals

$$\int_0^2\int_{-\pi/2}^{\pi/2}r^2\,rd\theta dr=\pi\int_0^2 r^3\,dr=4\pi$$