In class we tried to figure out the value (if any) of

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + 5y^2}$$

The limit along any straight line is zero, but the limit along the curve $y = x^3$ is $1/6$, so in this case the limit does not exist.

I was asked how you can show that the limit *does* exist, when it really does, and so I tried to illustrate by computing a very similar limit,

$$\lim_{(x,y)\to(0,0)} \frac{x^7 y}{x^6 + 5y^2}$$

I suggested switching to polar coordinates (and I will show later why that's a good idea) but I fumbled this one in class. What I should have done was to note that our function may be written

$$\frac{x^7 y}{x^6 + 5y^2} = \frac{x^6}{x^6 + 5y^2} \cdot xy \quad ;$$

the first factor is certainly less than one because the denominator is bigger (by $5y^2$) than the numerator. The other factor may be expressed in polar coordinates as $xy = r^2 \cos(\theta) \sin(\theta)$, which is no more than r^2. Therefore when we take the limit as (x, y) approaches (0,0), we must certainly have $r \to 0$, and thus in succession we learn that $r^2 \to 0$, thus $xy \to 0$, and thus $f(x, y) \to 0$.

But what about other functions of the form

$$\lim_{(x,y)\to(0,0)} \frac{x^n y}{x^6 + 5y^2} \quad ?$$

If this limit is zero for some n, then it will also be zero for any larger n; and conversely if the limit does not exist for some n, then it can't exist for any smaller n either. So now that we know the limit is zero for $n = 7$ we know it's also zero for $n = 8, 9, \ldots$ (and actually the very same argument works for $n = 6$ too). And since the limit does not exist for $n = 3$, it doesn't exist for $n = 0, 1, 2$ either.

But what about $n = 4$ and $n = 5$? These are trickier. It turns out the limit is zero when $n = 4$ (and thus it must be zero for $n = 5$ too). I will explain why using the idea of polar coordinates.

So let's let $f(x, y) = \frac{x^4 y}{x^6+5y^2}$. It's clear that as we approach the origin along either axis, the values of f are all zero, and thus those limits are zero too. So there are only two choices: either the limit of the function really is zero, or there's some path approaching the origin along which the values of the function don't tend to zero.

Let's try to be as devious as we can be, and find a way to sneak up on the origin without letting the values of f shrink to zero. Here's my idea: as we approach the origin our polar coordinate r will drop to zero as we pass through smaller and smaller circles around the origin. My suggestion is to choose the path that passes through these circles in such a way that every time we cross a circle we do so at the point of the circle where the function takes on its largest possible value.

For example, as we head to the origin, we will eventually pass through the circle of radius 1, where $x^2+y^2 = 1$. Where should our path cross this circle? I vote that we search for the point on the circle where f is largest. That's not too hard to find. Since we are on the circle, x^2 is the same as $(1-y^2)$ and thus the value of our function may be written as

$$f(x,y) = \frac{(1-y^2)^2 y}{(1-y^2)^3 + 5y^2} = \frac{y - 2y^3 + y^5}{1 + 2y^2 + 3y^4 - y^6}$$

Since (x, y) is a point on the unit circle we must have y between 1 and -1. Which of these values of y will make this function biggest?

You know from calc-1 how to answer this question: there must be a largest value somewhere, and it occurs at one of the endpoints or at a point where df/dy is either zero or fails to exist. It turns out that the best y is a solution of the equation $y^8-2y^6+15y^4+7y^2 = 1$ and its numerical value is approximately 0.339055284. And if that's my y coordinate, I'll take my x coordinate to be $\sqrt{1-y^2} = 0.9407704$. So I have found one point where I will draw my path on my way towards the origin: I'll have it pass through the point (0.339055, 0.9407704) on the unit circle. As I pass through that point, I find the value of f there to be approximately 0.2094418.

(You could also have parameterized this circle as the set of points where $x = \cos(t), y = \sin(t)$, then expressed f in terms of t, then looked for the value of $t \in [0, 2\pi]$ where f achieves its maximum. You will of course come to the same point, simply expressed in terms of polar coordinates.)

If I have any hope of showing you that the limit of this function is not zero, I will continue to draw more points on my path, looking for a path which keeps the function as big as possible. Really the steps are no different from what I just did: if I stay on a circle of radius r, then $x^2 = r^2 - y^2$ and the values of f are expressed in terms of y alone (and r), and my job is to choose the y (between $-r$ and r) which maximizes this. I set the first derivative equal to zero and solve for y. The y we need will be a root of the equation

$$y^8 - 2r^2y^6 + 15y^4 + (2r^6 + 5r^2)y^2 - r^8 = 0$$

I'll let you try some small values of r for yourself; what you'll discover is that the solutions are approximately $\sqrt{.2} \cdot r^3$ Then the corresponding values of x are $\sqrt{r^2-y^2} \approx \sqrt{r^2 - r^6/5} \approx r - r^5/10$ (where I just used a Taylor series in that last step!)

So I have discovered a curve which approaches the origin and tries harder than any other curve to keep the values of f away from zero. This curve is very near to the curve parameterized by

$$x = r - r^5/10, \qquad y = \sqrt{\frac{1}{5}}\; r^3$$

as $r \to 0$.

Well, how did we do? The value of the function f at points on this last curve is a very messy function of r but you can get to know how this function behaves near zero by computing its Taylor series: here are the first few terms:

$$f(x, y) = \sqrt{\frac{1}{5}} \cdot \left(\frac{1}{2} r - \frac{1}{20} r^5 - \frac{9}{400} r^9 + \ldots \right)$$

And now here's the punchline: the values of this function will clearly decrease to zero as $r \to 0$. So even on this "optimal" curve, the values of f decrease to zero. Therefore we know that along *any* curve that approaches the origin, the values of f will tend to zero, and thus

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0$$

I hope you can see how this process would work for any function: to prove that the limit of a function really is a number L, figure out the curve that approaches the origin by passing through the curves $r =$constant at the points (r, θ) where f is largest, and see what the limit along this curve is. Repeat with a different curve which passes through the points making f the smallest. If these two curves give the same limit, then any other curve must also tend to the same limit.

In practice, this is far too much work, and people look for slicker ways to prove the limit is zero, such as I did with

$$\frac{x^7 y}{x^6 + 5y^2}.$$

But as usual it's nice to have a universal method that can be applied when all the fast tricks seem to fall short!