Students:

Some of you asked for more details about why I don't use what is called the Cauchy Principal Value for improper integrals. I will explain by way of example.

Suppose I asked you to compute the definite integral

$$\int_{x=-2}^{x=+1} \frac{(10+4x)}{(5x+x^2)^3}\,dx$$

You should observe that the function fails to be continuous at $x = 0$ (and also at $x = -5$, but that does not concern us). So our recipe is to interpret the integral as the sum of two integrals

$$\int_{x=-2}^{x=0} \frac{(10+4x)}{(5x+x^2)^3}\,dx + \int_{x=0}^{x=+1} \frac{(10+4x)}{(5x+x^2)^3}\,dx$$

The two are handled similarly: as a limit of integrals with limits-of-integration slightly away from zero. In each case you can evaluate the integral using antiderivatives; it's easily done with the u-substitution, using $u = 5x + x^2$ so that $du = (5 + 2x)dx$:

$$\int \frac{(10+4x)}{(5x+x^2)^3}\,dx = \int \frac{2}{u^3}\,du = \frac{-1}{u^2} = \frac{-1}{(5x+x^2)^2}$$

So by the Fundamental Theorem of Calculus,

$$\int_{x=-2}^{x=0} \frac{(10+4x)}{(5x+x^2)^3}\,dx = \lim_{R\to 0^-} \frac{-1}{(5R+R^2)^2} + \frac{1}{36}.$$

That limit does not exist (it "equals $-\infty$") so our original integral does not converge. (The other integral, from $x = 0$ to $x = 1$, also diverges, in exactly the same way. But as soon as either of the two halves diverges, we say the original integral diverges.)

Now, sometimes people say "yes, but I want the 'Cauchy Principal Value' of the integral". That's the one that comes from *symmetrically* winnowing down on the singularity at $x = 0$: the Principal Value of the integral is

$$\lim_{a\to 0^+} \left(\int_{x=-2}^{x=-a} \frac{(10+4x)}{(5x+x^2)^3}\,dx + \int_{x=a}^{x=+1} \frac{(10+4x)}{(5x+x^2)^3}\,dx \right)$$

The first integral, as noted above, would equal $-1/(-5a + a^2)^2 + 1/6^2$ and similarly the second one would be $-1/6^2 + 1/(5a + a^2)^2$ Now remember, the "right" way to evaluate the integral is to compute the limits of these two separately, and then add; but both those limits do not exist. When people ask for the CPV they want you to add first, and THEN take the limit. The difference is that sometimes – as here – the two individual integrals get very large but are of opposite signs, so that adding them first leaves a small value, which may have a limit.

In our case the sum is (after some algebra)

$$\frac{-20}{a(a^2-25)^2}$$

and so as a decreases to zero, this becomes more and more negative; the Principal Value is "$-\infty$".

But here is why you need to be very careful with these things. You may have noticed that you used exactly the same u-substitution on both halves of the original integral; why not do both at once? That is, we simply substitute u for $5x + x^2$ to see that

$$\int_{x=-2}^{x=+1} \frac{(10+4x)}{(5x+x^2)^3}\,dx = \int_{u=-6}^{u=6} \frac{2}{u^3}\,du$$

This last integral is "obviously" zero, since we're integrating an odd function over a symmetric interval. More precisely you can compute its Cauchy Principal Value as the limit

$$\lim_{a\to 0^+}\left(\int_{u=-6}^{u=-a} \frac{2}{u^3}\,du + \int_{u=a}^{u=6} \frac{2}{u^3}\,du\right)$$

But those two proper integrals will exactly cancel out, so the limit as $a \to 0^+$ is zero.

To summarize: I computed the Principal Value of my original integral and I got ∞ ; then I used a u-substitution and computed the Principal Value of that, and got zero — a very different answer!

There is a very logical reason why this happened. The u-substitution replaced the singularity at $x = 0$ with a singularity at $u = 0$ — no problem. But it is not a "symmetric" u-substitution: the interval from $x = -a$ to $x = +a$ around the bad x-coordinate corresponds to the interval from $u = -5a + a^2$ to $u = +5a + a^2$ around the bad u-coordinate; that interval of x's is symmetric, but that interval of u's is not. So in a very natural way, the idea of using u-substitutions forces us to consider asymmetric intervals around singularities, which is contrary to the spirit of Principal Values.

So you have two ways out of this conundrum:

1. You can interpret all improper integrals as meaning the Principal Value; in that case you have to remember that u-substitutions cannot be used for improper definite integrals.

2. You can interpret all improper integrals as we did in class; in that case u-substitutions work fine but you have to accept that more integrals will have no defined value at all.

The first choice leads you to say that the value of the original integral is $-\infty$, and that the value of zero obtained by u-substitution is just plain wrong; the second choice leads you to say that the original integral simply does not have a value.

I have opted for the latter approach and I will ask you to do likewise in this course. If you ever need the Principal Value then compute it carefully! :-)