Here are some answers and comments about the free-response questions on the second M427L midterm.

For FR1, the integral evaluates to zero. But there are lots of ways to get an answer of zero by erroneous calculation, so I looked carefully at what you wrote.

It IS possible to compute this directly with Fubini's theorem, but realistically you would have to split the region into two (or four) pieces so that you could write out the limits of integration properly. Many of you wrote out double integrals which amounted to integrating the right function over a wrong domain (usually a triangle or parallelogram that *contains* the original region but is too big). I believe you would always get integrals of $\pm 61/3$ or $\pm 67/3$ over individual quadrants, and so the integrals over half-diamonds would be $\pm 128/3$. In the end they should all cancel.

What I expected instead, though, was for you to use the Change-of-Variables theorem. The presentation of the domain was supposed to make you think that $u = x + 2y$ and $v = x - 2y$ would make a terrific pair of coordinates for this particular problem. Inverting these formulas gives you the corresponding relationships $x = (u+v)/2, y = (u-v)/4$ which then define the transformation function $T : \mathbf{R}^2 \to \mathbf{R}^2$. The pairs (u, v) that are needed to provide our diamond are exactly the pairs in the square $[-2, 2] \times [2, -2]$ in the u, v plane. So that's (1) the re-writing of the domain of integration. Next is (2) the re-writing of the original integrand, which simplifies to $\frac{3}{4}(u^2 - v^2) + 16(u + v)$. Finally there's (3) the re-writing of the differential $dx\,dy = |\det(T')|\,du\,dv$. Depending on the order in which you list the input variables u and v, and the order of the output variables x,y, you can rearrange either the columns or the rows of the matrix T', and thus the determinant will be either $+1/4$ or $-1/4$. But you MUST remember that the fudge factor is the absolute value of this, so it's $+1/4$ no matter how you compute it. Anyway with (1),(2),(3) accomplished, you would probably just use Fubini's theorem to compute the integral as a double integral over the square. It comes to zero.

Other changes of variable are also reasonable: for example, you get the whole diamond by adding any positive fractions of the vectors $u = \langle 2, 1\rangle$ and $v = \langle 2, -1\rangle$ to the point $(-2, 0)$, so the parameterization $T(u, v) = (-2 + 2u + 2v, 0 + u - v)$ carries the square $[0, 1] \times [0, 1]$ to the diamond. Here $|\det(T')| = 4$ and $f \circ T$ is a bit messy but again the integral (of course) still evaluates to zero.

A couple of students were clearly concerned about getting the integral equal to zero. More of you should! I know I have encouraged you to think *first* about the value you expect from an integral (approximately), and *then* to compute it. In this case I would have thought about the integral as the sum of the integrals of $6xy$ and $32x$ separately. The latter has the property that it changes its sign as you replace x by $-x$, which means that the integral over the left- and right- sides will be negatives of each other. (Since there's no reference to y at all, the integrals over the top- and bottom- halves will be equal.) Similarly, the value of the first integral in just one quadrant will simply change sign as we cross *either* axis. So when we add together the values of two integrals over four quadrants, we get lots of pairs that cancel out, and the resulting integral is zero.

That's a lovely trick but PLEASE remember that it's not enough that the *regions* of integration have a symmetry across the axes; one can only make headway with a symmetry argument if one also knows that the *integrand* has this kind of symmetry or anti-symmetry.

For FR2, I really wanted students to compute this area as an integral:

$$\text{Area}(C) = \int_C 1\, dA$$

Of course in order to evaluate a surface integral, one must begin by parameterizing the surface C . Students used several different parameterizations of the cone, all of them great. In every case, it's useful to know that the part of the cone we care about ends where the cone meets the sphere. The x, y, z coordinates of such a point have x^2+y^2 equal both to z^2 and to $6z - z^2$, so for such points, $2z^2 = 6z$: (either $z = 0$ or) $z = 3$. So the surfaces meet within that plane, and the intersection obviously forms a circle, with $x^2 + y^2 = z^2 = 9$ — a circle of radius 3, at a height of 3.

The cone was described implicitly in Cartesian coordinates, but you can also present it explicitly. That is, you can think of the cone as the graph of the function $f(x, y) = \sqrt{x^2 + y^2}$; in that case you know the easiest parameterization of the cone is $T(u, v) = (u, v, f(u, v))$ where the pairs (u, v) range over the projection of the cone down to the x, y plane, which for us is the disk at the origin having a radius of 3. In such a setting we have seen that the Jacobian factor is always $\sqrt{1 + f_x^2 + f_y^2}$ and in this case that becomes a little bit messy but simplifies to the constant $\sqrt{2}$. So the area of the cone is the integral of this constant over the disk, which is then equal to $\sqrt{2}$ times the area of the disk, i.e. to $9\pi\sqrt{2}$. In case you didn't recognize that you just needed the area of the disk, you would have to integrate $\sqrt{2}$ over that disk: either use Fubini's Theorem (the limits of integration are -3 to $+3$ for x, say, and for fixed x, y varies from $-\sqrt{9 - x^2}$ to $+\sqrt{9 - x^2}$) or use the Change of Variables Theorem to evaluate this in polar coordinates (where $(r, \theta) \in [0, 3] \times [0, 2\pi]$).

Other students got a parameterization either directly from cylindrical coordinates or because they saw that a pattern like that would work. Each point in space has cylindrical coordinates (r, θ, Z) as well as Cartesian ones (x, y, z), and the relationship between the coordinate sets is that $x = r\cos(\theta), y = r\sin(\theta)$, and $z = Z$. It's then trivial to describe the cone implicitly in cylindrical coordinates: $x^2 + y^2 = z^2$ implies $r^2 = Z^2$ (just substitute the formulas) and so $Z = r$. So actually the surface has now been given *explicitly* in cylindrical coordinates, which allows us to trivially describe it *parametrically* in cylindrical coordinates too: just let $r = u, \theta = v$, and then Z will be the corresponding function of u and v, which in this case is just $Z = u$. Changing then back into Cartesian coordinates gives us our parameterization: $T(u, v) = (u\cos(v), u\sin(v), u)$. (If you're keeping track, we just went through 5 of the 6 corners of the "prism" that I described for you way, way back in the semester when we discussed the different ways to describe a surface in two coordinate systems!) Anyway now you're ready to compute the factor $||T_u \times T_v||$ and you can compute that it comes to $\sqrt{2}u$. You need to integrate that over the rectangle $[0, 3] \times [0, 2\pi]$ in the u, v plane.

Other students did something similar using spherical coordinates (ρ, θ, ϕ). Just as in the previous paragraph we use the translation formulas for x, y, z in terms of these 3 new coordinates to get an implicit description of the surface inspherical coordinates: it's $(\rho\cos(\phi))^2 = (\rho\sin(\phi))^2$, which simplifies to $\cos(\phi) = \sin(\phi)$, which means $\phi = \pi/4$, which you can see in the picture of the cone anyway, if you recall what it is that ϕ measures!

Anyway, this is actually an explicit description of the cone in this coordinate system, which gives an easy paramaterization $\rho = u, \theta = v, \phi = \pi/4$, and then we go back to Cartesian coordinates to get a parameterization of them: $x = u\cos(v) \cdot 1/\sqrt{2}, y = u\sin(v) \cdot 1/\sqrt{2}, z = u \cdot 1/\sqrt{2}$. This is very similar to the parameterization of the previous paragraph, the difference being that the factor u now is synonymous with ρ (the distance to the origin) rather than with r (the distance to the z-axis). This time $||T_u \times T_v||$ turns out to be just u, and this time the region of integration is the rectangle $[0, 3\sqrt{2}] \times [0, 2\pi]$.

I also had students who recognized this as a surface of revolution. You may have learned at some point how to compute the surface area of the surface that's obtained by spinning the graph of a function $y = f(x)$ around either the x axis or the y axis. Those are, in general, different surfaces — try it with the graph $y = x^2$ above the interval $[0, 1]$ — so they have different areas. In our case you can obtain a surface congruent to our cone by spinning the line segment $y = x$ (for $x \in [0, 3]$) around *either* axis. But rather than simply quoting a formula from Calculus-1 for computing such an area, let me observe that the spinning creates a parameterization of the surface, so we have another way to compute this surface integral. Each point $(x, f(x))$ in the graph of f spins into a circle in 3-space, which we can parameterize with a parameter t: if we spin around the x axis we get the points $(x, y, z) = (x, f(x)\cos(t), f(x)\sin(t))$, and if we spin around the y axis we get the points $(x, y, z) = (x\cos(t), f(x), x\sin(t))$. In either case, as x and t vary, we parameterize the whole surface. I will let you go through the whole $||T_u \times T_v||$ thing but by the time you have integrated over all $t \in [0.2\pi]$, you will be left with $\int f(x)\sqrt{1 + f'(x)^2}\, dx$ in the first case, and $\int x\sqrt{1 + f'(x)^2}\, dx$ in the other. (Some authors write these more elegantly as $\int y\sqrt{dx^2 + dy^2}$ and $\int x\sqrt{dx^2 + dy^2}$ respectively, which reveals the symmetry of the situation.) Anyway, in our case, with $f(x) = x$, these formulas obviously agree, and yield the correct surface area again.

There is one more way to compute the area of a cone: look up a formula. That's not really appropriate for this exam but it does give a correct answer of course. Even if you've never seen such a formula, I hope you have in your life constructed a cone out of paper: start with a disc of radius R and cut out a central wedge to leave a PacMan-like shape, and then just glue the straight edges together. If the resulting circle has a radius $r < R$, then that means that after you cut out the wedge, you left only a fraction r/R of the original circumference, and hence all that remains is the same fraction r/R of the original area. Since the original area was πR^2, that means the remaining area is $\pi r R$. When you're looking at the cone itself, r is the radius of the circle at its edge, and R is the distance from the edge points to the tip of the cone. In our example, those numbers are $r = 3$ and $R = 3\sqrt{2}$.

Quite a few people wanted to add in "the area of the base". I defy you to draw the figure I described in the statement of the problem and locate something to be called "the base". Presumably you're talking about the disk that's in the plane $z = 3$, but none of the points in the interior of that disc have Cartesian coordinates that satisfy my criterion $x^2 + y^2 = z^2$, so why do you want to add in the area of such points? (They're not on the sphere, either!) This also applies to the people who looked up the formula for surface area of a cone somewhere and were told it's $\pi r(r + \sqrt{r^2 + h^2})$. Please, people, think before you maniuplate formulas!