BENNETT LINEAR ALGEBRA PRIZE EXAM Dec 11 2019

1. Find five rational numbers z, y, x, w, v with the property that for every three numbers A, B, C we have

$$(A^5+B^5+C^5)-2(A^3+B^3+C^3)(A^2+B^2+C^2) = z\,S^5+y\,S^3T+x\,S^2U+w\,ST^2+v\,TU$$

where $S = A+B+C$, $T = AB+BC+CA$, and $U = ABC$. (You may assume that five such numbers exist.)

ANSWER: Assuming that there are five such numbers that work for any A, B, C we try some combinations of A, B, C to get five linear constraints on the variables which we will then solve. Here are some particularly simple examples

A	B	C	S	T	U	equation
0	0	1	1	0	0	$z = -1$
-1	2	2	3	0	-4	$243z - 36x = -207$
0	1	1	2	1	0	$32z + 8y + 2w = -6$
0	2	-1	1	-2	0	$z - 2y + 4w = -39$
1	1	-2	0	-3	-2	$6v = 42$
1	1	-1	1	-1	-1	$z - y - x + w + v = -5$
1	2	-2	1	-4	-4	$z - 4y - 4x + 16w + 16v = -17$
1	1	1	3	3	1	$243z + 81y + 9x + 27w + v = -15$

Taking the first five of these equations gives us a linear system to solve, represented by the augmented matrix

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -1 \\ 243 & 0 & 36 & 0 & 0 & -207 \\ 32 & 8 & 0 & 2 & 0 & -6 \\ 1 & -2 & 0 & 4 & 0 & -39 \\ 0 & 0 & 0 & 0 & 6 & 42 \end{array}\right)$$

I find the solution to be $(z, y, x, w, v) = (-1, 5, -9, -7, 7)$.

Remark: It is a theorem that any polynomial in multiple variables A_i which is symmetric (that is, the value of the polynomial is unchanged under any permutation of the variables) may be expressed as a polynomial in the *elementary symmetric polynomials* for those variables: those are the coefficients of the various powers of X in the product $(X + A_1)(X + A_2)\ldots$. In this problem I simply listed on the right side of the equation all the monomials in S, T, U which are of total degree 5 in A, B, C.

This idea can also be profitably used in problem 3.

2. Suppose $T : V \to V$ is a linear transformation on an n-dimensional vector space V such that the image of T is exactly the same as the kernel (nullspace) of T. Prove that n must be even.

ANSWER: Let $W = \text{Im}(T) = \text{Ker}(T)$, and let $\{b_1, b_2, \ldots, b_k\}$ be a basis for W. Since $W = \text{Im}(T)$, each b_i is the image $T(c_i)$ of some vector in V (not unique!). I claim that $\mathcal{B} = \{b_1, b_2, \ldots, b_k, c_1, c_2, \ldots, c_k\}$ is a basis for V, which will mean $n = 2k$ is even.

First we show $\mathcal{B}$ spans V. Given $v \in V$ we may find scalars x_i such that $T(v) = \sum x_i b_i$ because the b_i span $\text{Im}(T)$. But then $T(v) = \sum x_i T(c_i) = T(\sum x_i c_i)$, which means $v - (\sum x_i c_i)$ lies in $\text{Ker}(T)$, and that subspace is by hypothesis also spanned by the b_i. Thus there are other scalars y_i with $v - (\sum x_i c_i) = \sum y_i b_i$, which means v is indeed in the span of $\mathcal{B}$.

Next we show that $\mathcal{B}$ is a linearly independent set. Suppose that there were some scalars x_i and y_i such that $(\sum x_i c_i) + (\sum y_i b_i) = 0$. Apply T to both sides of this equation: the b_i all lie in $W = \text{Ker}(T)$ so we conclude $0 = \sum x_i T(c_i) = \sum x_i b_i$ But the b_i are linearly independent so all the x_i are zero. Thus our putative linear relation is simply $\sum y_i b_i = 0$, but again the independence of the b_i now forces all the y_i to be zero as well.

This argument recreates the idea behind the proof of the Rank-Nullity Dimension Theorem: the statement that for *any* linear transformation on a finite-dimensional vector space, we have

$$\dim(\text{Ker}(T)) + \dim(\text{Im}(T)) = \dim(V)$$

Of course if you know that theorem you may apply it directly to answer Question 2.

3. For a certain 3×3 matrix X we know the traces $\text{Tr}(X) = 0$, $\text{Tr}(X^2) = 42$, and $\text{Tr}(X^3) = -60$. Compute $\det(X)$.

ANSWER: The trace of a matrix X is both the sum of its diagonal entries and the sum of the roots of its characteristic polynomial P_X (which are the eigenvalues of X) counted according to their algebraic multiplicity. In our case there must be three roots r_1, r_2, r_3, whose sum is zero since $\text{Tr}(X) = 0$. Thus we have $r_3 = -r_1 - r_2$.

Now by diagonalizing X (or considering the Jordan Normal Form of X) we see that the roots of P_{X^2} are the squares of the roots of P_X, and similarly for P_{X^3}. So the other two data points tell us that

$$r_1^2 + r_2^2 + (-r_1 - r_2)^2 = 42 \qquad \text{and} \qquad r_1^3 + r_2^3 + (-r_1 - r_2)^3 = -60$$

With a bit of algebra we then have

$$r_1^2 + r_2^2 + r_1 r_2 - 21 = 0 \qquad \text{and} \qquad r_1 r_2 (r_1 + r_2) - 20 = 0$$

Multiply the first equation by r_1 and subtract the second to see that $r_1^3 - 21r_1 + 20 = 0$. This equation has three roots, 1, 4, and -5, which must then be the three roots of P_X. The determinant of X is then the product of these roots, which is -20.

4. Let $R : V \to V$ be a linear transformation on a finite-dimensional vector space V, and suppose $R^2 = I$. Show that for every vector $v \in V$ there exist a unique pair of vectors $v_1, v_2 \in V$ having $R(v_1) = v_1$, $R(v_2) = -v_2$, and $v = v_1 + v_2$.

ANSWER: Let $v_1 = \frac{1}{2}(I+R)v = \frac{1}{2}v + \frac{1}{2}R(v)$ and $v_2 = \frac{1}{2}(I-R)v = \frac{1}{2}v - \frac{1}{2}R(v)$. Clearly $v_1 + v_2 = v$.

We have $R(v_1) = \frac{1}{2}R(v) + \frac{1}{2}R^2(v) = \frac{1}{2}R(v) + \frac{1}{2}v = v_1$ and in the same way $R(v_2) = \frac{1}{2}R(v) - \frac{1}{2}R^2(v) = \frac{1}{2}R(v) - \frac{1}{2}v = -v_2$. So we have found one decomposition of v into parts with the desired properties.

Let us also prove uniqueness. Suppose $v = u_1 + u_2$ is another decomposition of v into summands with $R(u_1) = u_1$ and $R(u_2) = -u_2$. From $u_1 + u_2 = v_1 + v_2$ we conclude that $w_1 = u_1 - v_1$ and $w_2 = v_2 - u_2$ must be equal. Now, w_1 is fixed by R since u_1 and v_1 are, and likewise w_2 is negated by R since u_2 and v_2 are. But then if we apply R to both sides of the equation $w_1 = w_2$, we deduce $w_1 = -w_2$, so that $w_2 = -w_2$ and hence $w_2 = 0$. This in turn makes $w_1 = 0$, and thus $u_1 = v_1$ and $u_2 = v_2$. So in the end there is only one decomposition of the vector v with the desired properties.

There is no reason the vector space has to be finite-dimensional. In essence we are proving that there are enough eigenvectors to span the whole of V, the only possible eigenvalues being $+1$ and -1.

5. For a nonzero number c we define A_n to be the $n \times n$ matrix with $A_{ii} = 1$, $A_{i,i+1} = c$, and otherwise $A_{ij} = 0$. For example

$$A_4 = \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Find a matrix B with $BAB^{-1} = A^t$ (the transpose of A).

ANSWER: Let B be the matrix with $B_{ij} = 1$ if $i+j = n+1$ and $B_{ij} = 0$ otherwise. This matrix is invertible (indeed, it is its own inverse!) and I claim it has the desired property.

Rather than verify this by a matrix calculation, let us see how one could deduce this form for B; along the way we will see what other matrices B are valid answers.

Let $e_1, e_2, \ldots, e_n$ be the standard basis vectors in $\mathbf{R}^n$. The form of the matrix A_n shows that $A_n e_1 = e_1$ (i.e. e_1 is a $+1$-eigenvector for A_n) and then for $i > 1$ we have $A_n e_i = e_i + ce_{i-1}$. Similarly $A_n^t e_n = e_n$ and for $i < n$ we have $A_n^t e_i = e_i + ce_{i+1}$.

Now, we want an invertible matrix B with the property that $BA_n = A_n^t B$. It is sufficient to ensure that $BA_n e = A_n^t Be$ for each basis vector e. So we will decide what vector Be_i should be for each i in turn; that will fill in each of the columns of B.

For example when $i = 1$ we see that $BA_n e_1 = Be_1$ is supposed to equal $A_n^t Be_1$, which means Be_1 must be a $+1$-eigenvector of A_n^t. Thus we necessarily have $Be_1 = ke_n$ for some scalar k. (This k must be nonzero lest B have a kernel and thus not be invertible.)

Next $BA_n e_2 = B(e_2 + ce_1) = (Be_2) + cke_n$ is to equal $A_n^t Be_2$; that is, $v = Be_2$ must be a vector for which $A_n^t v = v + cke_n$. The vector ke_{n-1} has this property, so we will insist that $Be_2 = ke_{n-1}$. (It's actually not hard to show that the set of all vectors with this property are the vectors in the span of e_n and e_{n-1}. But we need only one.)

Continuing in this way, if we have already decided that $Be_{i-1} = ke_{n-i}$ then from $BA_n e_i = B(e_i + ce_{i-1}) = Be_i + cke_{n-1}$ we see that $v = Be_i$ must satisfy $A_n^t v = v + kce_{n-i}$; but our description of the action of A_n^t shows that $v = ke_{n-i-1}$ will suffice.

The matrix B with $Be_i = ke_{n+1-i}$ for every i is the scalar multiple k times the anti-diagonal matrix

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & \ldots & 1 & 0 & 0 \\ & & \ldots & & & \\ 0 & 1 & \ldots & 0 & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 \end{pmatrix}$$