BENNETT LINEAR ALGEBRA PRIZE EXAM Dec 10 2019

Name:____________________________ **UT EID:**________________________

Linear Algebra Course:________ **When?**________ **Instructor:**______________

Permanent Mailing Address:____

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E-mail address:____

College (Natural Sciences, Engineering, etc.)____________________________

Submit your solutions on the sheets provided, with your name on each sheet. No calculators allowed. You must justify your claims.

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1. Find five rational numbers z, y, x, w, v with the property that for every three numbers A, B, C we have

$$(A^5+B^5+C^5)-2(A^3+B^3+C^3)(A^2+B^2+C^2) = z\,S^5+y\,S^3T+x\,S^2U+w\,ST^2+v\,TU$$

where $S = A+B+C$, $T = AB+BC+CA$, and $U = ABC$. (You may assume that five such numbers exist.)

2. Suppose $T : V \to V$ is a linear transformation on an n-dimensional vector space V such that the image of T is exactly the same as the kernel (nullspace) of T. Prove that n must be even.

3. For a certain 3×3 matrix X we know the traces $\mathrm{Tr}(X) = 0$, $\mathrm{Tr}(X^2) = 42$, and $\mathrm{Tr}(X^3) = -60$. Compute $\det(X)$.

4. Let $R : V \to V$ be a linear transformation on a vector space V, and suppose $R^2 = I$. Show that for every vector $v \in V$ there exist a unique pair of vectors $v_1, v_2 \in V$ having $R(v_1) = v_1$, $R(v_2) = -v_2$, and $v = v_1 + v_2$.

5. For a nonzero number c we define A_n to be the $n \times n$ matrix with $A_{ii} = 1$, $A_{i,i+1} = c$, and otherwise $A_{ij} = 0$. For example

$$A_4 = \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Find a matrix B with $BAB^{-1} = A^t$ (the transpose of A).

Answers will soon appear at `http://www.math.utexas.edu/users/rusin/Bennett/`