1. Express the function $f(x) = \sin^3(x)\cos^2(x)$ as a linear combination of the functions $\sin(nx)$ and $\cos(nx)$, for $n = 0, 1, 2, \ldots$. (Possible hint: the decomposition should give a function which at the very least takes on the right values of f for a few well-known values of x.)

ANSWER: First let us establish some context. In the 19th century, Joseph Fourier realized that every periodic function f (meaning, a function which satisfies $f(x+2\pi) = f(x)$ for every real number x) could be expressed as a linear combination of the functions $\sin(nx)$ and $\cos(nx)$, for $n = 0, 1, 2, \ldots$. Informally, this set of trig functions forms a basis for the set of all period functions. But making clear sense of what exactly is being claimed is rather difficult and leads to the study of *Fourier Analysis*. For our purposes it is enough to know that our function has a (unique) such *Fourier expansion*.

It is actually easy to find this expansion without any Linear Algebra, if you know the right facts about trigonometry. The Angle Addition Formulas can be combined to give expressions for products such as $\cos(\alpha)\cos(\beta)$ in terms of sines and cosines of $\alpha \pm \beta$; several iterations of these can expand larger products of powers. The expansion can also be derived from DeMoivre's formulas linking the trigonometric and exponential functions, e.g. $\cos(x) = (e^{ix} + e^{-ix})/2$, which in particular suggests that we needn't use any $n > 5$. But let us solve the original question using Linear Algebra.

Two functions are equal if and only if they take the same values at the same points. So we might, for instance, look for a linear combination

$$g(x) = \sum_{n\geq 0} a_n \cos(nx) + \sum_{n\geq 1} b_n \sin(nx)$$

which agrees with the original function $f(x) = \sin^3(x)\cos^2(x)$ at many points x. We might use for example $x = 0$ and the points $x = \pm\pi/k$ ($k = 2, 3, 4, 6$) and $x = \pm 2\pi/3$; that would give 11 equation with which we might be able to solve for 11 unknowns ($a_0, \ldots, a_5$ and $b_1, \ldots, b_5$).

It's easy to be more efficient about this than writing an 11×11 matrix. To begin with, we need only ensure the functions f and g agree at all the positive points x_i: then since f is clearly an odd function, it can only be a linear combination of odd functions (the sines), meaning all the coefficients a_i must be zero; we simply want $f(x) = b_1 \sin(x) +$

$b_2 \sin(2x) + \ldots + b_5 \sin(5x)$. Evaluating at $x = \pi/k$ as above (and scaling each a bit for appearances) gives equations

$$\begin{array}{cccccc} b_1 & & -b_3 & & +b_5 & = 0 \\ b_1 & +b_2 & & -b_4 & -b_5 & = 3/16 \\ b_1 & -b_2 & & +b_4 & -b_5 & = 3/16 \\ b_1 & +\sqrt{2}b_2 & +b_3 & & -b_5 & = 1/4 \\ b_1 & +\sqrt{3}b_2 & +2b_3 & +\sqrt{3}b_4 & b_5 & = 3/16 \end{array}$$

The unique solution here is $b_1 = 1/8, b_2 = 0, b_3 = 1/16, b_4 = 0, b_5 = -1/16$ which suggests $\sin^3(x)\cos^2(x) = \frac{1}{8}\sin(x) + \frac{1}{16}\sin(3x) - \frac{1}{16}\sin(5x)$, which is the answer that can be obtained using trig identities.

Of course, this does not guarantee that $f = g$, merely that g agrees with f on a small set of values. To go further we can use the inner product that can be evaluated for any pair of periodic functions p_i by

$$\langle p_1, p_2 \rangle = \int_0^{2\pi} p_1(x) p_2(x)\, dx$$

Amazingly, the basis functions form an orthogonal basis with respect to this inner product, and thus we can recover the coefficients a_n and b_n in a Fourier expansion of any function $f(x)$ by simply computing the inner products

$$a_n = \langle f, \cos(nx) \rangle / \langle \cos(nx), \cos(nx) \rangle /$$

and likewise for the coefficients b_n.

2. An *Inverted Pascal's Triangle* (IPC) is an arrangement of 55 numbers in 10 rows; the nth row (counting from bottom up) has n numbers in it, and every entry in the IPC is the sum of the two numbers diagonally above it. If only some of the 55 cells have numbers in them and the others are left blank, it may be possible to determine what numbers belong in the empty cells, according to the rules. What is the smallest number of entries that must be filled in to enable us to determine the missing entries?

ANSWER: Let x_i be the number in the ith cell (numbered from the bottom up and from left to right). The an arrangement of any 55 numbers is indeed an IPC if the x_i satisfy a sequence of equations of the form $x_i = x_{i+n} + x_{i+n+1}$ where n is the row number of cell i:

$$x_1 = x_2 + x_3; \qquad x_2 = x_4 + x_5, x_3 = x_5 + x_6; \qquad x_4 = x_7 + x_8, \ldots$$

In our case, this is a collection of 45 homogeneous equations in 55 unknowns, meaning the set of IPCs is a subspace of $\mathbf{R}^{55}$, the nullspace of the corresponding 45×55 matrix M.

But it is clear that M has rank 45; indeed it is already row-reduced, with $M_{ii} = 1$ for all $i \leq 45$ and $M_{ij} = 0$ whenever $j < i$. Thus the set of IPCs is a 10-dimensional subspace of $\mathbf{R}^{45}$.

If we are presented with values for some of the cells, we effectively add equations of the form $x_i = c_i$ to the set (for various constants c_i); If the additional equations are inconsistent, of course there will be no solution to the puzzle of filling in the remaining entries. But if these equations are consistent, each one of them can reduce the dimension of the nullspace by at most 1 (depending on whether the added equations are independent or not). Therefore, in order to result in a puzzle with a unique solution, we must add at least 10 such equations (and they must form a set of equations that is both consistent and independent). Thus we certainly cannot expect a unique solution if fewer than 10 cells are filled in.

On the other hand, it is possible for an IPC with only 10 entries filled in to have a unique solution. For example, if (only) all the entries in the top row are filled in, we can simply propagate the values into the remaining cells by successively adding terms in pairs, and find the unique solution to that particular IPC puzzle.

Thus the correct answer to the original question is 10.

3. The set M_n of all $n \times n$ matrices with real entries forms a vector space. Find a basis of M_3 consisting of elements of M_3 that all commute with one another, or prove that no such basis exists.

ANSWER: M_n is an n^2-dimensional vector space and it is easy to find bases for it, but for $n > 1$ the basis cannot consist of pairwise-commuting matrices: if $\{A_i \mid i = 1, 2, \ldots, n^2\}$ is such a basis, then given any two matrices X and Y we would be able to find sets of scalars x_i and y_j with

$$X = \sum_{i=1}^{n^2} x_i A_i \qquad \text{and} \qquad Y = \sum_{j=1}^{n^2} y_j A_j$$

But then XY would be a double sum $\sum_i \sum_j (x_i y_j)(A_i A_j)$ and similarly we would have $YX = \sum_i \sum_j (x_i y_j)(A_j A_i)$; if every pair A_i and A_j commute, then these expressions would be identical and we would have $XY = YX$. But for any $n > 1$ there do exist pairs of non-commuting $n \times n$ matrices X and Y, so this would be a contradiction. Hence there can be no such basis.

4. There are 2^{16} four-by-four matrices whose entries are all 1's and 3's. Find the average of all their determinants.

ANSWER: The average is zero.

If $R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ then for any matrix M, RM is the result of swapping the first two rows of M (an elementary row operation). Since $R^2 = I$, we can pair off most of the 2^{16} matrices into pairs that are identical except for having their first two rows switched. However, this leaves unpaired the matrices M whose first two rows are equal ($RM = M$); there are 2^{12} of these.

So we will sum the determinants of all these matrices and divide by their number (2^{16}). The unpaired matrices all have a determinant equal to zero because they have a pair of equal rows. As for the other pairs of matrices, their determinants are negatives of each other ($\det(RM) = \det(R)\det(M) = -\det(M)$), so the sum of their determinants is zero. Hence the sum of all the determinants is zero, as is their average.

For what it's worth, just over half of these matrices are singular

5. Suppose that $L : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear transformation and that $\ker(L) = \{0\}$ (that is, the null space of L consists only of the zero vector in $\mathbf{R}^2$). First, show that there is a linear transformation $K : \mathbf{R}^3 \to \mathbf{R}^2$ such that $K(L(v)) = v$ for every $v \in \mathbf{R}^2$ (that is, $K \circ L = I_2$, the identity map on $\mathbf{R}^2$). Then decide which of the following is true:

(a) $L \circ K$ must equal I_3, the identity map on $\mathbf{R}^3$

(b) $L \circ K$ cannot equal I_3

(c) Whether $L \circ K = I_3$ or not depends on the choices of K and L.

ANSWER: Let e_1, e_2 be a basis for $\mathbf{R}^2$. Since $\ker(L) = 0$, the vectors $f_1 = L(e_1)$ and $f_2 = L(e_2)$ are linearly independent (in $\mathbf{R}^3$). Let f_3 be any vector outside the subspace spanned by f_1 and f_2; then $\{f_1, f_2, f_3\}$ is a basis for $\mathbf{R}^3$. In particular, we define linear transformations $K : \mathbf{R}^3 \to V$ (where V could be any vector space) simply by selecting any three vectors v_i in V and then defining $K(f_i) = v_i$ (and extending K to the rest of $\mathbf{R}^3$ by linearity).

Thus we may define a linear transformation $K : \mathbf{R}^3 \to \mathbf{R}^2$ simply be requiring $K(f_1) = e_1, K(f_2) = e_2$, and $K(f_3) = 0$. Then $K(L(e_1)) = e_1$ and $K(L(e_2)) = e_2$, and hence $K \circ L$ agrees with I_2 on this basis of $\mathbf{R}^2$, and hence (by linearity) on all of $\mathbf{R}^2$.

So we have constructed a "left inverse" for L. You know from Linear Algebra classes that if K and L are two linear maps on a *single* vector space V and V is *finite*-dimensional, then whenever $K \circ L = I_V$, it follows that $K \circ L$ must equal I_V as well. But that does not apply if K and L are maps between vector spaces of *different* dimensions, as in our case.

In fact, the dimension of the image of linear map can never be higher than the dimension of the domain; so since we constructed K to be defined on $\mathbf{R}^2$, this means the dimension of the image of K can be no more than 2. But then the image $L(K(\mathbf{R}^2))$ must be also at most 2. This means $L \circ K$ cannot be "onto" (cannot be a surjection, onto all of $\mathbf{R}^3$), and thus most definitely cannot equal I_3 which *is* a surjecton. So (b) is correct.