BENNETT LINEAR ALGEBRA PRIZE EXAM **Dec 07 2021**

Name:______________________________ **UT EID:**________________________

Linear Algebra Course:________ **When?**________ **Instructor:**____________

Permanent Mailing Address:____

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E-mail address:____

College (Natural Sciences, Engineering, etc.)____________________________________

Submit your solutions on the sheets provided, with your name on each sheet. No calculators allowed. You must justify your claims.

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1. Let V be the set of functions of the form $f(x) = p(x)\, e^{-x}$ where $p(x)$ is a polynomial of degree at most 4. Note that the derivative $D(f) = f'$ of every element of V is also a member of V. Find a basis $\mathcal{B}$ of V, and then find the matrix representation of D with respect to this basis. Is there another basis for which the matrix representing D is a diagonal matrix?

2. Suppose u and v are vectors in $\mathbf{R}^3$ and that we know these lengths: $||u|| = 3, ||u+v|| = 4$, and $||u - v|| = 6$. What is the length $||v||$ of the vector v?

3. Suppose A is a 2×2 matrix which satisfies $A^3 = A$. Show that A^2 must be equal to one (or more) of 0, I, A, or $-A$.

4. Recall that the *trace* $\mathrm{Tr}(M)$ of a real, $n \times n$ matrix M is the sum of the diagonal entries of M.

(a) Find such a matrix B for which $\mathrm{Tr}(B^2) < 0$

(b) Show that if C is symmetric then $\mathrm{Tr}(C^2) \geq 0$

(c) Show that if M has n distinct real eigenvalues then $\mathrm{Tr}(M^2) \geq 0$

5. Find the rank, and a basis for the null space (=kernel), of the $n \times n$ matrix M whose (i, j) entry is $M_{ij} = (i + j - 2)^2$. For example, for $n = 4$ the matrix M is

$$\begin{pmatrix} 0 & 1 & 4 & 9 \\ 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \end{pmatrix}$$

Answers will soon appear at `http://www.math.utexas.edu/users/rusin/Bennett/`