BENNETT DIFFERENTIAL EQUATION PRIZE EXAM May 10 2022

1. Austin entrepreneur Melon Tusk plans to build an amusement park ride built on an extremely tall tower – its height will be several times the diameter of Earth! Customers will rocket to the top of the tower and then descend inside a spherical capsule in free-fall until retrorockets brake the capsule shortly before returning to land. Until those braking rockets fire, the capsule will accelerate according to Newton's Law ($F = ma$); the only forces acting on the sphere will be gravitational force (whose strength is inversely proportional to the square of the distance to the center of the earth) and air resistance (which is proportional to the square of the velocity). With those assumptions, will the falling sphere achieve a terminal velocity?

ANSWER: No; the speed of fall will grow unbounded until additional force is applied by the retrorockets.

If y represents the height of the falling sphere from the center of the earth, then y will be a function of time; its time-derivative y' will be the (upward) velocity and y'' the (upward) acceleration. So the premise of the problem is that y'' is a constant multiple of the net force, which in turn is the sum of two parts: a negative multiple of $1/y^2$ and a positive multiple of $(y')^2$. In other words, $y(t)$ will satisfy a differential equation of the form $y'' = -A/y^2 + B(y')^2$.

If the object achieves a terminal velocity v_0, then y' will be constant and y'' will be zero. But then the differential equation forces $-A/y^2 + Bv_0^2 = 0$, i.e. $y = (A/B)/v_0^2$, which implies the height is constant. This in turn forces $v_0 = 0$, which is inconsistent with the differential equation. So no, a terminal velocity is impossible.

But something similar can occur. Since the equation is autonomous (i.e. time t is not involved except as the base of the derivatives) we may use the given differential equation to find the relationship between height y and velocity $v = y'$, or better, between y and kinetic energy $E = v^2/2$, since

$$\frac{dE}{dy} = \frac{dE}{dt}\Big/\frac{dy}{dt} = v\frac{dv}{dt}\Big/v = y'' = -\frac{A}{y^2} + Bv^2 = 2BE - \frac{A}{y^2}$$

To visualize the graph of E versus y, note that the curve where $y^2E = A/(2B)$ is the set of points where dE/dy can vanish; to the right (and up), E will increase with y, and to the left (and down) E decreases as a function of y. Thus the graph of any solution $E(y)$ of this differential equation will be concave up, with a smooth section in which E hardly varies as y does. (As *time* increases, height decreases, so the capsule's state in the E, y plane will follow this curve from right to left.) If we start with a high enough speed (relative to our height), then air resistance will predominate and slow us down to a minimum speed (and minimum value of E); after air resistance is minimal, we will continue to fall (with near-constant speed) so gravitational attraction will increase over time, causing speed to increase again as we move higher on the E axis. In short, while there is no *terminal* velocity, there can be (if we start far out in the E, y plane) a single moment of zero acceleration, and near that moment our velocity is *nearly* constant for a while.

(Actually, the given assumptions are not especially realistic. The force of air resistance certainly increases with velocity, although (depending on the aerodynamics of the capsule design) it may be more appropriate to assume that force is proportional to a lower power of y' such as $(y')^{4/3}$. More significantly, for a fall of this magnitude, we must account for the change in air density as altitude increases. If for example we assume the force of air resistance also drops in proportion to the square of the altitude, then a terminal velocity would definitely be achieved.)

2. Find the general solution of the differential equation $(y')^3 - 27xy' + 27y = 0$. Is it possible to find two different solutions that have the same initial conditions $y(x_0) = y_0$ and $y'(x_0) = v_0$?

ANSWER: If we differentiate the given ODE we obtain the further information that

$$(3y'^2 - 27x)(y'') = 0$$

thanks to a fortuitous cancellation of terms. This means we must either have $y'' = 0$ (which is to say, y is a linear function $y = Ax + B$) or $y'^2 - 9x = 0$. This latter equation is easily integrated as well: the general solution is $y = \pm 2x^{3/2} + C$.

Now, if we return to the original differential equation we find constraints on these constants A, B, C: in the linear case we must have $A^3 + 27B = 0$ and in the nonlinear case we must have $C = 0$. So we may present the complete solution set as the family of curves

$$y^2 = 4x^3 \quad \text{and} \quad y = (3r)x - (r^3) \quad \text{for any } r \in \mathbf{R}$$

Note in particular that both functions $y = 2x^{3/2}$ and $y = 3x - 1$ pass through the point $(x_0, y_0) = (x_0, f(x_0)) = (1, 2)$ with a slope of $f'(x_0) = 3$, so the answer to the final question is “yes”. You know that in general the solution to a first-order ODE is uniquely determined by the initial conditions, but that statement is actually a theorem, which has hypotheses that must be met, and in this example they are not!

This equation is an example of a *Clairaut Equation*; all equations of this type have a particular solution not included in the parameterized family of solutions.

3. Find the general solution of the system of differential equations

$$\frac{dy}{dx} = y - z - 4, \qquad \frac{dz}{dx} = y + 3z + 4x$$

ANSWER: The problem may be presented in matrix form as $v' = Av + w$ where $v(x)$ and $w(x)$ are the vector-valued function $v(x) = (y(x), z(x))$ and $w(x) = (-4, 4x)$.

We may attempt to diagonalize the matrix A, computing its characteristic polynomial $\lambda^2 - 4\lambda + 4$, but this has only one root, $\lambda = 2$, and we discover only a one-dimensional space of eigenvectors, the multiples of $u_1 = (-1, 1)$. So the matrix A is not diagonalizable, but may be put into its Jordan Normal Form using u_1 and $u_2 = (1, 0)$ as a basis. That is, we have found matrices P and $Q = P^{-1}$ with

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, QAP = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Then the equation $v' = Av + w$ implies $(Qv)' = (QAP)(Qv) + (Qw)$, that is, if Qv is the vector-valued function $(f(x), g(x))$, then

$$g'(x) = 2g(x) + (4x - 4) \qquad f'(x) = 2f(x) + g(x) + (4x)$$

With the components now decoupled, we may solve first for g then for f as solutions of first-order linear ODEs. The integrating factor in each case is e^{-2x} and so we quickly discover $g(x) = 1 - 2x + C_1 e^{2x}$, and then $f(x) = (-x - 1) + C_1 x e^{2x} + C_2 e^{2x}$

Finally computing $v = P(Qv) = P \cdot (f(x), g(x))^t$ allows us to solve for

$$y = (2 - x) + (C_1 - C_2)e^{2x} - C_1 x e^{2x}, \quad z = (-x - 1) + C_1 x e^{2x} + C_2 e^{2x}$$

for arbitrary constants C_1 and C_2.

4. Which solutions of the equation $f'(x) = x - f(x)^3$ stay bounded as $x \to +\infty$?

ANSWER:

None of them. If for some constant C we know $f(x) \leq C$ for all x in some interval $[T, \infty)$ then we would have $f'(x) \geq x - C^3$ on that interval, and so $f(x) = f(T) + \int_T^x f'(t)\,dt \geq f(T) + \int_T^x (t - C^3)\,dt = \frac{1}{2}x^2 - C^3 x + D$ for some constant D. This is obviously unbounded for large x.

It is possible to deduce much more qualititative information about the solutions $f(x)$, without ever giving a formulaic description of them. For example, The curve C defined by $x = y^3$ splits the plane into two regions. If the graph of f passes through any point $(x_0, y_0) = (x_0, f(x_0))$ in the upper region, then $y_0^3 > x_0$ there, so that $f'(x_0) = x_0 - f(x_0)^3 < 0$ and the function is decreasing at that point. Thus the graph of f is decreasing throughout the upper region, until the graph of f crosses this curve C into the lower region.

By a similar analysis, the graph of f will be increasing at any point in the lower region. But we can show that the graph will never again cross C into the upper region. Indeed, at points just below C the slopes $f'(x)$ of the graph will not only be positive but will be increasing, as $f''(x) = 1 - 3f(x)^2 f'(x) = 1 - 3f(x)^2(x - f(x)^3)$ will be positive if y is sufficiently close to $x^{1/3}$. (Having $y^3 > x - (1/3)x^{-2/3}$ will do, for example.) Since the slopes anywhere *on* C are zero, they cannot be positive just to the left and below C.

So the solution curves may start above C, reach a minimum y-coordinate on C, become increasing and concave up on a short interval, then continue to increase but become concave down on the rest of the real line, yet (because the solution is not bounded) never approach a horizontal asymptote and (because they cannot cross C) never even growing as large as $x^{1/3}$ again.

5. Find a non-constant function $u = u(x, y)$ which satisfies

$$y\, u_{xy} + u = 0$$

at all points in the first quadrant, and vanishes along the curve $y = e^{-x/2}$.

ANSWER: Since this equation is linear in u, we may use Separation of Variables, seeking solutions which are linear combinations of products $u(x, y) = f(x)g(y)$ (with neither f and g being identically zero). Such product solutions require $yf'(x)g'(y) + f(x)g(y) = 0$, i.e. $yg'(y)/g(y) = -f(x)/f'(x)$; this is only possible if the two sides are a constant c. This in turn forces f to be a constant multiple of $e^{-x/c}$ and g to be a constant multiple of y^c. That is, our product solutions are constant multiples of functions $y^c e^{-x/c}$. Any linear combination of such functions is also a solution to the PDE.

Now, the condition that u vanish along a curve restricts the linear combinations we may use. On the curve $y = e^{-x/2}$, the basis function $y^c e^{-x/c}$ takes values equal to $e^{(-c/2-1/c)x}$. Since the functions e^{kx} with different values of k are linearly independent, we cannot expect a linear combination to vanish unless we use different values of c for which the exponents $-c/2 - 1/c$ are equal. If c_1 is one of the values, then we may take $c_2 = 2/c$ for the other — it turns out these are the only possibilities. In that case our linear combination has the form

$$Ay^c e^{-x/c} + By^{2/c} e^{-cx/2}$$

whose value on the curve $y = e^{-x/2}$ is simply $(A + B)e^{-dx}$ (for $d = 1/c + c/2$), which vanishes iff $A = -B$.

Taking for example $c = 1$ and $A = 1$ gives the function

$$u(x, y) = ye^{-x} - y^2 e^{-x/2}$$