1. Find the sums of each of the following series. Simplify your answers.

(i) $\sum_{n=0}^{\infty} \left(\frac{x-1}{x}\right)^n$ where $x > 1$

(ii) $\sum_{n=0}^{\infty} (-1)^n \frac{(\tan^{-1}(x))^{2n}}{(2n)!}$

ANSWER: The first is a geometric series which converges (if $r = (x-1)/x = 1-(1/x)$ is less than 1 in magnitude) to $1/(1-r) = x$. This applies in particular when $x > 1$, since then $0 < 1/x < 1$, making $0 < r < 1$.

In the second series we recognize the form of the Maclaurin series for the cosine function,

$$\cos(u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!}$$

which converges for all real (or complex!) values of u. In particular, the given series will converge for all x to $\cos(\tan^{-1}(x)) = 1/\sqrt{1+x^2}$.

2. Compute the following limits

(i) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^3}{n^4} \cos\left(\pi \frac{k^2}{n^2}\right)$

(ii) $\lim_{n\to\infty} n \int_2^{2+3\sin(1/n)} x^{-2} e^x \, dx$

ANSWER: These two limits are simply instances of the very definitions of derivatives and integrals!

The sum in the first limit is a Riemann sum associated with an integral. More precisely, the Riemann integral $\int_a^b f(x)\,dx$ can be defined precisely as the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k) \frac{(b-a)}{n}$$

where the points x_k are located in the kth interval of width $(b-a)/n$ between a and b. In the special case where $a = 0$ and $b = 1$, the points $x_k = k/n$ fit this description, and then the given limit matches the above definition of the integral if $f(x) = x^3\cos(\pi x^2)$ for all $x \in [0,1]$. Therefore, the given limit is the definition of

$$\int_0^1 x^3 \cos(\pi x^2)\, dx$$

The antiderivative $\int x^3\cos(\pi x^2)\,dx$ may be evaluated with the help of Integration By Parts and the substitution $u = x^2$; one antiderivative is

$$\frac{x^2\sin(\pi x^2)}{2\pi} + \frac{\cos(\pi x^2)}{2\pi^2}$$

Thus with the Fundamental Theorem of Calculus we evaluate the integral (and thus the original limit) as $-1/\pi^2$.

The second limit is the definition of the derivative

$$F'(a) = \lim_{h\to 0} \frac{F(a+h)-F(a)}{h}$$

applied to the example $F(u) = \int_2^{2+3\sin(u)} x^{-2}e^x\,dx$ with $a = 0$, where h is written as $1/n$; then $F(a) = F(0) = 0$ and $F(a+h) = F(1/n)$.

(More precisely, *if* the limit (as $h \to 0$) exists, then the limit as $n \to \infty$ exists as well and the two are equal. The converse is not necessarily true: the stated problem would ordinarily be assumed to consider the limit only over *positive integer* value of n; the values of the integral for such values of n do not force the values for non-integral values of $n = 1/h$ to behave nicely, and even if they did, we would gain no information about what happens for *negative* h near 0.)

So our limit equals $F'(0)$, if this derivative exists, where $F(u)$ is as above. Well, this F may be viewed as a composite $F = f \circ g$ where

$$f(v) = \int_2^v x^{-2}e^x\,dx \qquad \text{and} \qquad g(u) = 2 + 3\sin(u)$$

so by the Chain Rule, $F'(0) = f'(2)\cdot g'(0)$. Obviously $g'(u) = 3\cos(u)$ so $g'(0) = 3$. On the other hand the Fundamental Theorem of Calculus shows $f'(v) = v^{-2}e^v$ for all $v > 0$ (FTC requires that the integrand be continuous on the interval between 2 and v) so in particular $f'(2) = e^2/4$.

We conclude that limit (ii) evaluates to $\frac{3}{4}e^2$.

The numerical value of this expression is approximately

$$5.5417920741979876704228\,20$$

The integral may be computed numerically for specific values of n to compute the terms of the sequence; the 1000th term is approximately

$$5.54179530379634658843170 8$$

so this suggests we have computed the limit correctly.

3. Compute the indefinite integral

$$\int \frac{1}{1 - x^{1/5}} \, dx$$

ANSWER: Under the substitution $x = u^5$ this integrand becomes a rational function: we get $\int (5u^4)/(1-u)\, du$. This integral can be evaluated easily using "long division" or perhaps more readily with another substitution $u = 1 - v$, yielding $\int -5(1 - 4v + 6v^2 - 4v^3 + v^4)/v dv = -5(\ln(|v|) - 4v + 3v^2 - (4/3)v^3 + (1/4)v^4) + C$. With a bit of algebra this may be re-cast in terms of u and then of x. The final result is

$$-\ln(|1 - x^{1/5}|) - 5x^{1/5} - \frac{5}{2}x^{2/5} - \frac{5}{3}x^{3/5} - \frac{5}{4}x^{4/5} + C$$

4. Find the volume of the solid torus (donut) obtained by rotating the unit disc $x^2 + y^2 \leq 1$ about the line $x + y = 6$.

ANSWER: There is a formula for the volume of a torus (in terms of its two radii) but we can compute this using integrals. We will simplify this only to the extent that, since we know the volume does depend only on the radii, we will rotate the figure so that the line $x+y=6$ is replaced by a vertical line at the same displacement from the disk, namely $x = 6\sqrt{2}$.

The the volume is easily computed by the "Method of Washers": $V = \int_{-1}^{+1} A(y)\, dy$, where $A(y)$ is the area of the slice of the torus having a given y coordinate. Each such slice is clearly an annulus (a "washer") centered on our vertical line. Its inner and outer radii are the distances between that central point $(6\sqrt{2}, y)$ and the points on the unit circle that share this y coordinate, namely $(\pm\sqrt{1-y^2}, y)$; that is, the inner and outer radii of the annulus are $6\sqrt{2} \pm \sqrt{1-y^2}$. Then the cross-sectional area $A(y)$ is $\pi R^2 - \pi r^2$. Writing the radii as $a \pm b$ for brevity, we see $R^2 - r^2 = (a+b)^2 - (a-b)^2 = 4ab$ and so the area of the annulus at height y is

$$A(y) = 4\pi \cdot 6\sqrt{2} \cdot \sqrt{1-y^2}$$

It follows that the volume of the solid torus is

$$24\sqrt{2}\pi \int_{-1}^{+1} \sqrt{1-y^2} = 12\sqrt{2}\pi^2$$

(The last integral is easily computed by recognizing it as computing the area of the upper half of a disk of radius 1.)

The "Method of Shells" may also be used.

5. There are 4 lines which are tangent to both of the circles $x^2 + (y-3)^2 = 1$ and $x^2 + (y+5)^2 = 4$. Find the equation of one of the lines.

ANSWER: More generally, if a point (a, b) is on a circle defined by the equation $x^2 + (y-c)^2 = r^2$ then of course $a^2 + (b-c)^2 = r^2$ but more importantly, the line tangent to this circle at this point has a slope

$$m = \frac{dy}{dx} = -\frac{dF/dx}{dF/dy} = -\frac{x}{y-c} = -\frac{a}{b-c}.$$

Since the tangent line passes through (a, b) it is not hard then to deduce that the tangent line satisfies the equation

$$ax + (b-c)(y-c) = r^2$$

(This line has the right slope and passes through the original point.)

On the other hand, there is a purely geometric way to determine something about a line tangent to *two* circles. Draw such a line, along with the line joining the circles' centers. Draw also the lines joining the points of tangency to the centers of their own circles. This creates two right triangles with a common angle at the end, hence the triangles are similar, the ratio of similarity being the ratio of the legs (which are radii). With a proportionality argument one can then determine the coordinates of the point where the line of (joint) tangency meets the line joining the centers. In our example, for instance, the x-axis is the line joining the centers, 8 units apart. The radii are in a 2:1 ratio, so the joint tangency lines that cross between the circles split the 8-unit distance in a 2:1 ratio as well, meaning they both pass through the point $(0, 1/3)$. The outer lines of joint tangency by contrast create two right triangles with hypotenuses of some lengths L and $L+8$, also in a 2:1 ratio, so $L = 8$, putting their intersection at the point $(0, 11)$.

So for example the point $(0, 11)$ satisfies the equation of a tangency line, which we have already shown is of the form $ax + (b-c)(y-c) = r^2$ for some point (a, b) on the top circle (where $c = 3$ and $r = 1$). Thus $8(b-3) = 1$ meaning $b = 25/8$. Then $a^2 + (b-c)^2 = r^2$ forces $a = \sqrt{63}/8$. (Either square root works.) Hence the line(s) must be $\sqrt{63}x + y = 11$. (The two choices of square root give mirror-image lines of tangency. The points of tangency on the lower circle are $(\sqrt{63}/4, -19/4)$.)

Similarly the point $(0, 1/3)$ only satisfies the equation $ax + (b-3)(y-3) = 1^2$ if $b = 21/8$, which requires $a = \sqrt{55}/8$, making the line(s) be $-\sqrt{55}x + 3y = 1$.