Here are some possible answers to the questions on the

Albert A. Bennett Calculus Prize Exam — Dec 5 2010

Note: It is quite possible that the same answers can be obtained by other methods. Indeed, creative means to derive correct answers are valued in mathematics generally and in this contest in particular.

This version corrects a small error in problem 4.

1. Evaluate $\lim_{x\to\infty} \dfrac{\sqrt{x^3 - x^2 + 3x}}{\sqrt{x^3} - \sqrt{x^2} + \sqrt{3x}}$.

Solution: Dividing top and bottom by $\sqrt{x^3}$ we get $\lim_{x\to\infty} \dfrac{\sqrt{1 - x^{-1} + 3x^{-2}}}{1 - \sqrt{x^{-1}} + \sqrt{3x^{-2}}}$. Both numerator and denominator now tend to 1, and so does the function.

Remark: This question was lifted from a 1998 Foxtrot cartoon :-)

2. Determine whether these series converge or diverge.

(a) $\sum_{n=2}^{\infty} \dfrac{n^8 - 1}{n^9 - 1}$

Solution: We may use the Limit Comparison test with the Harmonic series:

$$\lim_{n\to\infty} \frac{(n^8-1)/(n^9-1)}{(1/n)} = \lim_{n\to\infty} (n^9 - n)/(n^9 - 1) = \lim_{n\to\infty} (1 - n^{-8})/(1 - n^{-9}) = 1$$

so since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, so does our series.

(b) $\sum_{n=2}^{\infty} \dfrac{1}{\ln(n!)}$

Solution: Since $n!$ is the product of n integers no larger than n, we have $n! < n^n$. Therefore $\ln(n!) < n \ln n$ and so the series dominates $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$. But this latter series is easily seen to diverge, using the integral test:

$$\int_2^T \frac{1}{x\ln(x)}\, dx = \ln(\ln(x))\Big|_2^T$$

and so as $T \to \infty$, the integral diverges.

Remark: This question appeared on an old Putnam Exam!

3. Compute $\lim_{x\to 0} \dfrac{\cos(2x) + 2\sin(x^2) - 1}{x^4}$.

Solution: We can compute the Taylor series of the numerator as far as the x^4 term by recalling that $\sin(x) = x - x^3/3 + \ldots$ and $\cos(x) = 1 - x^2/2 + x^4/24 + \ldots$. Thus the numerator has a Taylor expansion of $(1 - 2x^2 + (2/3)x^4 + \ldots) + (2x^2 - x^6/3 + \ldots) - 1$ which is $(2/3)x^4 + \ldots$. Divide by x^4 and let $x \to 0$ to get a limit of $2/3$.

Remark: You may certainly use four iterations of L'Hopital's Rule — *if* you have the patience and algebraic skill ...

4. The four points $A = (-6, -2, 3)$, $B = (-6, 8, 3)$, $C = (-7, 5, 3)$, $D = (4, -6, 5)$, are all equally far from a point P. Find P.

Solution: P will lie on the plane of points equidistant from any pair. For example, the points equidistant from A and B lie on a plane perpendicular to vector $AB = \langle 0, 10, 0\rangle$ and pass through the midpoint $(A+B)/2 = (0, 3, 0)$. This plane is of the form $0x+10y+0z = c$ where necessarily $c = 30$. Likewise the plane equidistant from B and C is found to be $x+3y = 13$ and the plane equidistant from C and D is $11x-11y+2z = -3$. So the point P is the intersection of these three planes, i.e. the solution to a system of 3 equations in 3 unknowns: $x = 4, y = 3, z = -7$. ($P = (4, 3, -7)$ is of distance 15 from each of A, B, C, D.)

Remark: It is also natural to begin with three equations in the three unknown coordinates x, y, z of P: equations which say

$$\mathrm{dist}(P, A) = \mathrm{dist}(P, B) = \mathrm{dist}(P, C) = \mathrm{dist}(P, D).$$

Square each and subtract $x^2 + y^2 + z^2$ from all sides to get the same three linear equations shown above.

5. Compute the minimum value of the function

$$f(u, v) = \left(u - v\right)^2 + \left((3 - u) - \left(\frac{5}{v}\right)\right)^2$$

on the region where $v > 0$.

Solution: The function measures the square of the distance between the two points $P = (u, 3 - u)$ and $Q = (v, \frac{5}{v})$. The former is a point on the line $x + y = 3$ and the latter is a point (in the first quadrant) on the hyperbola $xy = 5$, so all we want to know is the minimum distance between these two curves. By symmetry, or a glance at the graph, it is clear that the points of closest approach are on the line $y = x$, which then must clearly be the points are $P = (3/2, 3/2)$ and $Q = (\sqrt{5}, \sqrt{5})$, so the value of the function is $\sqrt{2}(\sqrt{5} - 3/2)$.

Remark: The problem can also be treated as an unconstrained minimization problem by setting $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both equal to zero. The algebra is rather messy, though.

Moreover, this problem may be treated as a constrained minimization problem: we wish to minimize $(u - v)^2 + (U - V)^2$ subject to the constraints $u + U = 3$ and $vV = 5$. With the obvious choice of notation we then let $F = F_0 - \lambda_1 F_1 - \lambda_2 F_2$; the optimal point must satisfy $F_u = F_U = F_v = F_V = F_{\lambda_1} = F_{\lambda 2} = 0$ — six equations to solve for the six unknowns $u, U, v, V, \lambda_1, \lambda_2$. The equations are fairly simple and may be solved easily to get the points P and Q noted above.

This was adapted from Putnam Exam question B4, 1976