ALBERT A. BENNETT CALCULUS PRIZE EXAM – Dec 4 2011
Here are some possible responses to this semester's Bennett exam.

1. Let $f(x) = e^{-x}\sin(x^3)/x$ and $g(x) = \ln(1+e^{-x})$. Compute

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}.$$

Answer: The quotient is $\frac{\sin(x^3)}{x} \cdot \frac{e^{-x}}{\ln(1+e^{-x})}$. The first factor tends to zero because the numerator is never more than 1 (the "Squeeze Theorem"). The second factor tends to 1: as $x \to \infty$, $u = e^{-x} \to 0$, but as $u \to 0$, $u/ln(1+u) \to 0$ (as can be checked easily with L'Hôpital's Rule, for example). Thus the original limit is zero.

Note: Both f and g tend to zero, but L'Hôpital's Rule is *not* helpful if applied directly:

$$f'(x)/g'(x) = \left(\sin(x^3)/x + \sin(x^3)/x^2 - 3x\cos(x^3)\right) \cdot (1+e^{-x})$$

The second factor tends to 1, and in the first factor the first two summands tend to 0. But the last summand oscillates with increasing amplitude, and so $f'(x)/g'(x)$ does not have a limit at all. (You might want to read carefully what the "L'Hôpital's Rule" theorem actually says!)

2. For what values of a does this (improper) integral converge?

$$\int_a^\infty \frac{1}{\sqrt{|x^3(x-1)|}}dx$$

(*Possible Hint:* One approch uses the substitution $u = \frac{1}{x}$.)

Answer: Except over the interval [0,1], we may ignore the absolute value signs. In particular we can evaluate the integral easily whenever $a \geq 1$. Using the hint provided, we see that for $a > 1$,

$$\int_a^T \frac{1}{\sqrt{x^3(x-1)}}dx = \int_{1/T}^{1/a} \frac{1}{\sqrt{1-u}}du = -2\sqrt{1-u}|_{1/T}^{1/a}$$

and now taking the limit as $T \to \infty$ we see that for all $a > 1$ the original integral converges (to $2(1-\sqrt{1-(1/a)})$).

The integral also converges when $a = 1$: we need only consider the limit of the previous integrals as $a \to 1^+$, for which the value increases to 2.

For $0 < a < 1$ we may then consider the integral over $[a, \infty)$ to be the sum of the integrals over $[a, 1]$ and $[1, \infty)$, the latter of which is now convergent. The former may be treated similarly: for $a > 0$ we may write the integrand as

$$\int_a^T \frac{1}{\sqrt{x^3(1-x)}}dx = \int_{1/T}^{1/a} \frac{1}{\sqrt{u-1}}du = 2\sqrt{u-1}|_{1/T}^{1/a}$$

and as we let T increase to 1 from the left, this expression converges (to $2\sqrt{(1/a)-1}-2$).

However, as we let a decrease to 0, $1/a \to \infty$ and then the value of this last integral diverges to $+\infty$. So the improper integral does NOT converge when $a = 0$ (and then obviously not for any $a < 0$ either). So the correct answer is that a should lie in the interval $(0, \infty)$.

3. Does the series

$$(\frac{-1}{1}) + (\frac{1}{2} + \frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9}) + \ldots$$

converge? This series can also be written

$$\sum \frac{\varepsilon(n)}{n}, \quad \text{where} \quad \varepsilon(n) = \begin{cases} -1, & \text{if } n \text{ is a perfect square} \\ +1, & \text{otherwise} \end{cases}$$

Answer: No, this series diverges. In the nth cluster there are $2n-1$ terms

$$\frac{1}{n^2-2n+2} + \frac{1}{n^2-2n+3} + \ldots + \frac{1}{n^2-1} - \frac{1}{n^2}$$

The net sum of the last two is positive, and each of the others is larger than $1/n^2$, so the total value of this set of terms is larger than $(2n-3)/n^2$. For all $n > 1$ this is in turn at least as big as $n/n^2 = 1/n$, so the sum of all these groups of terms is then larger than $\sum_{n>1}(1/n)$ which diverges. (It is the harmonic series.)

4. Compute the first six terms of the Taylor series for $\sec(x)$, that is, determine the coefficients $a_0, \ldots a_5$ in the expansion

$$\sec(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots$$

Answer: Since $\sec(x) = 1/\cos(x)$ for all x, and the Taylor series for the cosine is

$$1 - \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{6!} + \ldots$$

it is sufficient to compute an expansion for

$$\frac{1}{1 - \frac{x^2}{2} + \frac{x^4}{24} + \ldots}$$

But (again an easy Taylor series computation)

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \ldots$$

which we can use with $u = \frac{x^2}{2} - \frac{x^4}{24}$; only a few terms involve powers of x below the 6th; we get

$$\sec(x) = 1 + \left(\frac{x^2}{2} - \frac{x^4}{24}\right) + \left(\frac{x^2}{2}\right)^2$$

This will give the required terms of the series. (The series can be continued in the same way; the first few terms are

$$\sec(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \ldots)$$

Note: it is very inefficient to use the all-purpose formula $\sum \frac{f^{(n)}(a)}{n!}(x-a)^n$ here; for example the fifth derivative of secant is

$$\sec(x)\tan(x)^5 + 58\sec(x)^3\tan(x)^3 + 61\sec(x)^5\tan(x)$$

5. Find the point on the paraboloid $z = 2x^2 + y^2$ which is closest to the plane $6x + 4y + z + 3 = 0$.

Answer: There was a typo in the statement of the problem: this particular plane intersects the paraboloid because there are pairs (x, y) where $2x^2 + y^2 + 6x + 4y + 3 = 0$ (namely any point on the ellipse $2(x+3/2)^2+(y+2)^2 = 11/2$). So any point $(x, y, 2x^2+y^2)$ is on the paraboloid and of distance zero from the plane.

What had been intended was for the coefficient "3" to have been a "9"; indeed the same answer is obtained for any coefficient "D" greater than 17/2:

The distance from (x_0, y_0, z_0) to the plane $Ax + By + Cz + D = 0$ is

$$\frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

so for points on our paraboloid that distance is $\frac{|6x+4y+(2x^2+y^2)+D|}{\sqrt{6^2+4^2+1^2}}$. In order to minimize this we need only find the positive maximum or negative minimum of $6x+4y+(2x^2+y^2)+D$. This occurs where the gradient vanishes: $6 + 4x = 4 + 2y = 0$, i.e. at the point $(x, y) = (-3/2, -2)$ (where $z = 17/2$).