1. The equation $x^y = y^x$ describes a curve in the first quadrant of the plane containing the point $P = (4, 2)$. Compute the slope of the line that is tangent to this curve at P.

ANSWER: Take logarithms to see that this equation is equivalent to $\ln(x)/x = \ln(y)/y$, that is, $f(x) = f(y)$ where $f(x) = \ln(x)/x$.

Now use Implicit Differentiation to get $f'(x) = f'(y)\frac{dy}{dx}$, i.e.

$$\frac{dy}{dx} = \frac{f'(x)}{f'(y)} = \frac{(1 - \ln(x))/x^2}{(1 - \ln(y))/y^2}.$$

At the point P this gives a value of

$$\frac{dy}{dx} = \frac{(1 - 2\ln(2))/16}{(1 - \ln(2))/4},$$

about -0.31.

If you know it, you may also use the Implicit Function Theorem, which tells us directly that on the curve $F(x, y) = 0$ we have

$$\frac{dy}{dx} = -\left(\frac{\partial F}{\partial y}\right)^{-1}\left(\frac{\partial F}{\partial x}\right)$$

You could also use Implicit Differentiation directly on the equation $x^y = y^x$; it is helpful to recall that when a is independent of x, we have $(d/dx)(a^x) = a^x \cdot \log(a)$.

The graph of this function f is very revealing: $f(x)$ is positive iff $x > 1$, increases to a maximum of $1/e$ at $x = e$, and then decreases to zero as $x \to \infty$. Thus f achieves each value in $(0, 1/e)$ precisely twice. Hence $f(x) = f(y)$ iff $x = y$ or else x and y are such a pair of numbers where f attains a single value — one of them in $(1, e)$ and the other in (e, ∞). Note that as $x \to 1^+$, $y \to \infty$, and vice versa.

So the graph of $x^y = y^x$ consists of the line $y = x$ and this set of pairs (x, y) with $f(x) = f(y)$. It has asymptotes at $x = 1$ and at $y = 1$ and is contained in the region where $x > 1$ and $y > 1$. The formula above for dy/dx then shows dy/dx is everywhere negative, so the graph is everywhere decreasing.

2. Determine whether this series is convergent or divergent:

$$\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(\ln(n))}}$$

ANSWER: The series diverges.

Each term is positive, and may be written as $e^{-(\ln(\ln(n))^2}$. Now, for all $x > 0$ we have $\ln(x) < \sqrt{x}$. (This is easily proved by noting that $f(x) = \sqrt{x} - \ln(x)$ has a minimum value at $x = 4$, where $f(4) = 2\ln(e/2) > 0$.) So $(\ln(x))^2 < x$ for all $x > 0$, including when $x = \ln(n)$, i.e. $e^{(\ln(\ln(n)))^2} < e^{\ln(n)} = n$. (Equivalently, let $y = \ln(\ln(n))$ and then note $(\ln(n))^{\ln(\ln(n))} = (e^y)^y = e^{(y^2)} < e^{(e^y)} = n$.) Thus the nth term of our series is greater than $1/n$, and hence this series diverges by comparison to the Harmonic Series.

(For comparison, $\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}$ converges, this one being the sum of $1/n^{\ln(\ln(n))}$, and that exponent is more than $1.019 > 1$ for all $n > 15$.)

3. Compute $\displaystyle\int_0^{\pi/4} \frac{1}{\cos(x) + \sin(x)}\, dx$.

ANSWER: The integral will be computed to be $\ln(\sqrt{2}+1)/\sqrt{2}$ by the Fundamental Theorem of Calculus. We may use any of several substitutions to help us find the antiderivative.

Simplest, perhaps, is to note that

$$\cos(x) + \sin(x) = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)\cos(x) + \sin\left(\frac{\pi}{4}\right)\sin(x)\right) = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$$

so if we use the substitution $u = x - (\pi/4)$ our integral becomes

$$\int_{u=-\frac{\pi}{4}}^{u=0} \frac{du}{\sqrt{2}\cos(u)} = \frac{1}{\sqrt{2}}\int_{u=-\frac{\pi}{4}}^{u=0} \sec(u)\, du = \frac{1}{\sqrt{2}} \ln\left(|\sec(u) + \tan(u)|\right)\Big|_{-\pi/4}^{0}$$
$$= -\frac{1}{\sqrt{2}}\ln(\sqrt{2}-1) = \frac{1}{\sqrt{2}}\ln(\sqrt{2}+1)$$

(More generally any linear combination of $\sin(x)$ and $\cos(x)$ may be combined into a single term $A\cos(x+B)$ in a similar way.)

A closely-related suggestion would be $u = \sin(x) - \cos(x)$; then $u^2 = 1 - 2\sin(x)\cos(x)$ and $dx/(\cos(x)+\sin(x)) = du/(2-u^2)$, which we must integrate from $u = -1$ to $u = 0$, to get the same conclusion.

Or we may use the half-angle substitution $u = \tan(x/2)$. I will let you check the trigonometry that proves

$$\cos(x) = \frac{1-u^2}{1+u^2}, \qquad \sin(x) = \frac{2u}{1+u^2}, \qquad dx = \frac{2\,du}{1+u^2}$$

This transforms our integral into

$$\int_{u=0}^{u=\sqrt{2}-1} \frac{1+u^2}{(1-u^2)+(2u)} \frac{2\,du}{1+u^2} = -2\int_{u=0}^{u=\sqrt{2}-1} \frac{du}{(u-1)^2-2}$$

which may be computed using Partial Fractions. (This half-angle substitution converts *any* rational function of the six trig functions into a rational function of u, which may then be computed using Partial Fractions.)

Or we may use deMoivre's formulas: $\cos(x) = (e^{ix} + e^{-ix})/2$ and $\sin(x) = (e^{ix} - e^{-ix})/(2i)$. In this case we may let $u = e^{ix}$ (so $du = iu\,dx$) and the substitution results in a rational function of u, again amenable to Partial Fractions.

Or, multiply numerator and denominator by $\cos(x) - \sin(x)$ to write the integrand as

$$\frac{\cos(x)}{1-2\sin^2(x)} + \frac{-\sin(x)}{2\cos^2(x)-1}$$

(Each denominator equals $\cos^2(x) - \sin^2(x)$.) Substitute $u = \sin(x)$ resp. $u = \cos(x)$, to transform each of the two indefinite integrals to $\pm\int \frac{du}{1-2u^2}$, which can be evaluated using Partial Fractions and then rewritten again in terms of sines and cosines. (Some care must be taken when evaluating this antiderivative at $x = \pi/4$ precisely because the extra factor $\cos(x) - \sin(x)$ which we introduced vanishes at this value of x, that is, we have created an improper integral. In that case we take a limit as the upper limit of integration approaches $\pi/4$ from the left, using L'Hôpital's Rule in the antiderivative.)

This last approach illustrates another general technique useful for trig integrals. What we did here was to write the integrand $f(x)$ as the sum of its even part $(f(x) + f(-x))/2$ and its odd part $(f(x) - f(-x))/2$; the first is a function $f_1(\cos(x))$ of $\cos(x)$ alone and the second may similarly be written as $\sin(x)\, f_2(\cos(x))$. For the second integral we use the substitution $u = \cos(x)$ to get $\int f_2(u)\, du$, that is we have removed the trigonometry. For the first integral, we first write f_1 in terms of *its* even and odd parts, one being a function $f_3(\cos^2(x))$ of $\cos^2(x)$ alone and the other being of the form $\cos(x) f_4(\cos^2(x)) = \cos(x) f_4(1-\sin^2(x))$; of these two parts the first is improved with the double-angle formula for cosine, and the second is amenable to the substitution $u = \sin(x)$.

4. A *wedding ring* is the three-dimensional solid that remains after drilling a cylindrical hole through the center of a sphere. Compute, with proof, the volume of metal in a metallic wedding ring that is 6mm tall when it rests on a table, as a function of the radius r of the hole that has been drilled.

ANSWER: When $r = 0$, we have drilled out none of the sphere, so the height is the diameter of the sphere, and the volume is the volume of a sphere of radius 3mm, namely $V = \frac{4}{3}\pi r^3 = 36\pi$ mm^3. Amazingly, the volume of metal is 36π mm^3 *irrespective of the value of* r*!*

Let s be the radius of the sphere, i.e. the distance from a point on the outside of the wedding ring to the center of the sphere. If that point is at a distance d from the central axis of the wedding ring, and at a height h above the center of the sphere, then a small right triangle will show that $s^2 = h^2 + d^2$. In the special case that the point is along the top edge of the wedding ring, this shows $s^2 = 3^2 + r^2$.

Now, this "wedding ring" can be viewed as a solid of revolution; we are spinning a region shaped like a thin letter D around the central axis. Thus the cross-section perpendicular to the central axis at a height of h is an annulus ("washer") with an inner radius of r and an outer radius of d, and hence of area $\pi d^2 - \pi r^2 = \pi(s^2 - h^2) - \pi(s^2 - 3^2) = \pi(9 - h^2)$.

Thus the volume of the ring is, as advertised,

$$\int_{h=-3}^{h=+3} \pi(9 - h^2)\, dh = \pi \left(9h - \frac{h^3}{3} \right) \Big|_{-3}^{3} = 36\pi.$$

One may also use the method of shells: at a distance t from the central axis ($0 < t < s$) the half of the figure lying above the central plane is a cylinder of height $\sqrt{s^2 - t^2}$ (and radius t), hence the total volume of the top half of the ring is $\int_r^s 2\pi t\sqrt{s^2 - t^2}\, dt = (2\pi/3)(s^2 - r^2)^{3/2}$, and yet as noted above $s^2 - r^2 = 9$mm^2, giving the same conclusion as above. One obtains precisely the same integral by computing the volume as an integral $2\int_A \sqrt{s^2 - (x^2 + y^2)}\, dx\, dy$ over the annulus A having radii r and s, when using polar coordinates.

This is a well-known question.

5. The curve parameterized by $x(t) = \cos^3(t)$, $y(t) = \sin^3(t)$, $z(t) = \cos(2t)$ passes through the point $(1, 0, 1)$ when $t = 0$ and passes through the point $(0, 1, -1)$ when $t = \pi/2$, having traversed a path of length $5/2$. (You don't have to prove this.) What point will it pass through after having traversed a length of exactly 1 ?

ANSWER: The length of a segment of a parameterized curve can be viewed as the distance traveled, and distance is the integral of speed (over time); speed is the magnitude of the velocity vector, and velocity is the rate of change of position. That is, the length of the curve between $t = a$ and $t = b$ is

$$L = \int_{t=a}^{t=b} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

In our case $x'(t) = -3\cos^2(t)\sin(t)$ and $y'(t) = 3\sin^2(t)\cos(t)$, so already we have $(x')^2 + (y')^2 = (3\sin(t)\cos(t))^2\left(\cos^2 t + \sin^2 t\right) = 9\sin^2(t)\cos^2(t)$. Next, since $z'(t) = -2\sin(2t) = -4\sin(t)\cos(t)$, we then have $(x')^2 + (y')^2 + (z')^2 = (\sin(t)\cos(t))^2(3^2 + 4^2)$, so the integrand is the square root of this: $5\,|\sin(t)\cos(t)|$. In our case we will have $t \in (0, \pi/2)$, so both the sine and the cosine are positive and the absolute value bars unnecessary. Thus

$$L = \int_{t=0}^{t=b} 5\sin(t)\cos(t)\, dt = \frac{5}{2}\sin^2(t)\Big|_0^b = \frac{5}{2}\sin^2(b)$$

This is equal to 1 when $\sin(b) = \sqrt{2/5}$. Then $\cos(b) = \sqrt{3/5}$ and $\cos(2b) = \cos^2(b) - \sin^2(b) = 1/5$, so the point we will have reached is

$$(x, y, z) = \left((3/5)^{3/2},\ (2/5)^{3/2},\ 1/5\right).$$

Of course there is nothing special about making the length equal to 1; we will have traversed any distance $\ell < 5/2$ when $\sin^2(b) = 2\ell/5$ so that we will be at position

$$p(\ell) = \left(\ (1 - \frac{2\ell}{5})^{3/2},\ (\frac{2\ell}{5})^{3/2},\ 1 - \frac{4\ell}{5}\ \right)$$

This function p provides a new parameterization of the same curve, with now the additional feature that for all t, the curve from $p(0)$ to $p(t)$ has arclength t, that is, we are traversing the curve at a constant speed of 1. This kind of re-parameterization is always possible (at least in principle, if not with formulas) for any smooth curve; it is called *parameterization with respect to arclength*.