ALBERT A. BENNETT CALCULUS PRIZE EXAM **Dec 09 2020**

1. Find the limit of the sequence $\{a_1, a_2, a_3, \ldots\}$ where

$$a_n = \left(\frac{(n+1)^{n+1}}{n^n} - \frac{n^n}{(n-1)^{n-1}} \right)$$

ANSWER: The limit is e.

Notice first that the terms of the sequence have the form $a_n = b_n - b_{n-1}$ where $b_n = \frac{(n+1)^{n+1}}{n^n} = (n+1)\left(1+\frac{1}{n}\right)^n$. You may know that the second factor $c_n = (1+\frac{1}{n})^n$ approaches e as $n \to \infty$, but that is not quite enough here. Even if you knew, for example, that $|c_n - e| < 2/n$ for every n (which *is* true), that would be enough to conclude that the numbers c_n do converge to e, but would also allow for the possibility that $c_n = e + (-1)^n/(n+1)$. Then it would follow that $b_n = (n+1)e + (-1)^n$, in which case we would find that $a_n = e + 2(-1)^n$, which doesn't even *have* a limit as $n \to \infty$. So it is necessary for us to determine not only the limit of the c_n, but to know *how quickly* they approach their limit.

So let's do this carefully. You know that $c_n = e^{n \ln(1+\frac{1}{n})}$. Use the Taylor theorem (with remainder) on the logarithm function to see that $\ln(1+(1/n)) = (1/n) - (1/2n^2) + E$ where the error term in the Taylor series will be positive but smaller in magnitude than $1/3n^3$. So what we must exponentiate is equal to $1 - (1/2n) + E'$ where $0 < E' < 1/3n^2$. Thanks to the laws of exponents and the fact that the exponential function is increasing, this tells us that $c_n = e^1 e^{-(1/2n)} E''$, where $E'' = e^{E'}$ lies between 1 and $e^{1/3n^2}$. Using Taylor series again, now for the exponential function, we learn that $e^{-(1/2n)} = 1 - (1/2n) + R$ where $0 < R < 1/8n^2$, and similarly $e^{1/3n^2} = 1 + S$ where S is "small". (Since $n \geq 1$, we can show that $0 < S < e^{1/3}/(2n^2)$; the $e^{1/3}$ comes from the maximum value of the first derivative of e^x on the interval $[0, 1/3]$.) So we have learned $c_n = e(1 - 1/2n + R)(1 + S)$ where both R and S are "small", specifically they are no larger than a multiple of $1/n^2$; the language for this is that both R and S are "$O(1/n^2)$". In this language, then, we have learned that $c_n = e(1 - (1/2n)) + O(1/n^2)$. Multiply now by $n+1$ to obtain a precise estimate for b_n: it is equal to $(n+1)e - e/2 + O(1/n)$

Replacing n by $n-1$ then shows $b_{n-1} = ne - e/2 + O(1/n)$. Subtracting gives $a_n = e + O(1/n)$ and in particular as $n \to \infty$, we see $a_n \to e$.

2. There is a function f whose graph lies in the first quadrant. When the graph is rotated around the x-axis we obtain a surface S which has the following unusual feature: the volume of the region that is inside S and lies between the planes $x = 0$ and $x = b$ is b^2, for every constant $b > 0$. What is the function f ?

ANSWER: For any three-dimensional shape, the volume of the portion of that shape lying between two planes $x = a$ and $x = b$ is $\int_a^b A(x)\,dx$ where $A(x)$ is the area of the cross-section of points that have a particular x-coordinate. A shape that is created by spinning a graph around the x-axis has circular cross-sections, of area $A(x) = \pi r^2$, the radius r being the value $f(x)$ of the function at that value of x. Applied to our present problem, this tells us that for any $b > 0$ we have

$$\int_0^b \pi f(x)^2\,dx = b^2$$

Now apply the Fundamental Theorem of Calculus: we may differentiate both sides with respect to b and discover that $\pi f(b)^2 = 2b$, so that the function must be $f(x) = \sqrt{2x/\pi}$.

3. Compute an antiderivative: $\displaystyle\int \frac{2\tan(x)}{\sqrt{1-\sin^4(x)}}\,dx$

ANSWER: Several times in this solution we involve a square root as a factor in an expression. By definition, real square roots are non-negative, which means $\sqrt{a^2} = |a|$, not a, but we will simplify without the absolute value bars, with the understanding that (depending on the range of our variables) our formulas may be off by a sign.

With that understanding, we have $\sqrt{1-\sin^2(x)} = \cos(x)$ and so the denominator is $\sqrt{1-\sin^4(x)} = \cos(x)\sqrt{1+\sin^2(x)} = \cos(x)\sqrt{2-\cos^2(x)}$ while the numerator is $2\sin(x)/\cos(x)\,dx$; thus the substitution $u = \cos(x)$ seems natural and gives the new antidifferentiation problem

$$\int \frac{-2\,du}{u^2\sqrt{2-u^2}}$$

We can get rid of the square roots with a further substitution $u = \sqrt{2}\cos(t)$: now that last numerator is $-2du = 2\sqrt{2}\sin(t)\,dt$ and the denominator becomes $2\sqrt{2}\cos^2(t)\sin(t)$. Then the integral is simply $\int \sec^2(t)\,dt = \tan(t)$ (plus arbitrary constants, of course).

To convert back to previous variables, write this as $\sqrt{\sec^2(t)-1} = \sqrt{(2/u^2)-1} = \sqrt{2-u^2}/u$. In terms of the original variable this is $\sqrt{2-\cos^2(x)}/\cos(x)$.

The answer may be presented differently, e.g. as $\sqrt{1+2\tan^2(x)}$, and there are other strings of substitutions that will lead to equivalent answers. (Students found many!)

(Check your answer by computing the derivative of this function. You will obtain a formula which *appears* to be correct when tested hastily with various trigonometric and algebraic identities. However, because of the square roots, this derivative is actually of the wrong sign for some x, namely the same values of x that make $\cos(x) < 0$. So as it turns out a correct antiderivative is $\sqrt{2-\cos^2(x)}/|\cos(x)|$.)

4. Let $f(x) = 21/(x^2+x+1)$ so that $f(2) = 3$ and $f(4) = 1$. For any value of d between 1 and 3, the horizontal line $y = d$ crosses the graph of f exactly once to form a region R_d bounded by this horizontal line, the graph, and the vertical lines $x = 2$ and $x = 4$. (You might call it a "butterfly" or "bow-tie" shape.) For what value of d is the area of R_d smallest?

ANSWER: If we do this cleverly, no lengthy calculation is involved! Our f is differentiable everywhere, with $f'(x) = -21(2x+1)/(x^2+x+1)^2$, which is negative for all $x > -1/2$. So we have a strictly decreasing, differentiable function f on an interval $[a, b]$ (in our case, $a = 2$ and $b = 4$). That's all we need to know about f ! For then any line $y = d$ will meet the graph at exactly one point (c, d) which has $c \in [a, b]$. (This c is the unique number in this interval for which $f(c) = d$.) Then the area of R_d is given by

$$\int_a^b |f(x) - d|\, dx = \int_a^c (f(x) - d)\, dx + \int_c^b (d - f(x))\, dx$$

This is easily evaluated using the Fundamental Theorem of Calculus, if you know an antiderivative F of f: it is simply $(F(c) - F(a) - d\cdot(c-a)) + (d\cdot(b-c) - (F(b) - F(c)))$ which simplifies to $2F(c) - F(a) - F(b) - d(2c - a - b)$.

Now, as we vary d trying to minimize this area, the values of c will change as well; indeed, we may phrase this problem as the question of how to choose the number c in the interval $[a, b]$ which minimizes the function $A(c) = 2F(c) - F(a) - F(b) - f(c)(2c - a - b)$. It is easy to differentiate this A since we know $F' = f$: with the product rule we get

$$\frac{dA}{dc} = (2f(c)) - (2f(c) + f'(c)\cdot(2c - a - b)) = -2f'(c)\left(c - \frac{a+b}{2}\right)$$

Then since $f'(c) < 0$ for every c, this derivative $A'(c)$ is negative when $c < (a+b)/2$ and positive when $c > (a+b)/2$. Thus the values of the area function $A(c)$ grow larger whether we move away to the left or to the right from the midpoint $(a+b)/2$ of the interval. Hence it is for c precisely at the midpoint that the area function A is minimized.

Thus the optimal horizontal line is the one with $d = f(\frac{a+b}{2})$ which in our case is $d = f(3) = \frac{21}{13}$.

In this problem it would have been possible for you to compute that antiderivative F.

$$F(x) = 14\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$$

but this calculation is onerous and, as shown above, unnecessary. (Sometimes it's easier to solve a more general problem than it is to work out just one example!)

5. Caasi Notwen is traveling around the first quadrant of the x, y plane; the x- and y-coordinates of her position at time t are denoted $x(t)$ and $y(t)$, respectively. She starts at the point (1,1). She notices that at each moment t her velocity maintains a rigid relationship to her position: the x- and y- components of her velocity are given, respectively, by

$$x'(t) = x(t)\,(-6 + 2y(t)) \qquad \text{and} \qquad y'(t) = y(t)\,(7 - 3x(t))$$

At how many different points can Caasi cross the line $y = 1$?

ANSWER: It's clear from the differential equation that the lines $x = 7/3$ and $y = 3$ split the first quadrant into four regions: as Caasi passes through the lower-left region, for example, she will have $x' < 0$ and $y' > 0$, so she will head roughly northwest; if and when she crosses the line $y = 3$ she will start traveling northeast, and so on. So it appears her path is a sort of spiral. But does it spiral in, or spiral out, or close back in on itself?

Caasi's path will be a curve in the plane which at each time t will have a slope equal to

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{y(t)}{x(t)}\frac{(7 - 3x(t))}{(-6 + 2y(t)}$$

Thus her path is the graph of a solution $y = y(x)$ to the differential equation

$$\frac{dy}{dx} = \frac{y(7 - 3x)}{x(-6 + 2y)}$$

This is a separable differential equation, so we separate the variables to rewrite it as

$$\frac{-6 + 2y}{y}\,dy = \frac{(7 - 3x)}{x}\,dx$$

which is easily integrated: $-6\ln(y) + 2y = 7\ln(x) - 3x + C$; the fact that her path takes her through the point (1,1) tells us that $C = 5$.

Now, we are asked about the number of values of x for which the point $(x, 1)$ lies on this curve, but in fact we can similarly answer the question for any horizontal line $y = b$. The curve will cross that line at each point (x, b) for which $7\ln(x) - 3x = -6\ln(b) + 2b - 5$, so we are simply asking how many solutions there are to an equation $7\ln(x) - 3x = k$ for certain constants k. That's easy to answer: just graph the function $g(x) = 7\ln(x) - 3x$ and see how many times it crosses a particular horizontal line. The slope $g'(x) = 7/x - 3$ is positive for $x < 7/3$ and negative for $x > 7/3$, so the graph increases and then decreases; it's also clear from the formula that g tends to $-\infty$ as $x \to 0+$ and as $x \to +\infty$. So it's clear from a picture that there are two solutions to the equation $g(x) = k$ when $k < g(7/3)$, and none when $k > g(7/3)$. Caasi started at one point that lay on the line $y = 1$, so there is precisely one more she could ever reach. (Since she started at the point with $x = 1$, and $1 < 7/3$, the other possible x coordinate will have $x > 7/3$.)

Which are the values of k that we will actually face in this analysis? When we introduced this variable k it was shorthand for $-6\ln(b)+2b-5$ where b is the $y-$coordinate of the line we hope to intersect. The analysis of the previous paragraph can be used for this function of b as well: its values decrease from $+\infty$ (when b is near 0) to a minimum which occurs at $b = 6/2 = 3$, and then rises to $+\infty$ as b increases to $+\infty$. In other words, as we ask about all the horizontal lines $y = b$, we will run the analysis of the previous paragraph for every value of k larger than $-6\ln(3) + 2\cdot 3 - 5 = 1 - \ln(729)$, in fact encountering each of these values of k twice (once for a $b < 3$ and again for a $b > 3$). Since in the preceding paragraph we saw that there are no points on the curve when k is too large, this will mean we will only have points on the curve for horizontal lines $y = b$ for values of b that are bounded both above and below.

Piecing this information together gives a nice view of the curve: it forms a closed loop with its highest and lowest points on the line $x = 7/3$ and with its leftmost and rightmost point on the line $y = 3$; the other coordinates of these extreme points can then be found using the curve's implicit equation $-6\ln(y) + 2y = 7\ln(x) - 3x + 5$. (I find them to be approximately (7/3, 0.6436), (7/3, 8.3224), (0.5758, 3), and (6.0728, 3) .)

Precisely these equation arise in the study of predator-prey relationships between two species, whose populations $x(t)$ and $y(t)$ change over time in relation to the present populations of both the predator species and the prey species.