ALBERT A. BENNETT CALCULUS EXAM ANSWERS 5/9/09

1. Find the largest possible volume for a right circular cone inscribed in a sphere of radius one. Recall that the volume of a right circular cone is $1/3\,hA$ where h is the height and A is the area of the base.

ANSWER: A cone of maximal volume would have its central axis run through the center of the sphere — otherwise a parallel translation would allow the cone to grow taller. By the same reasoning, a maximal cone would touch the sphere at its tip and base; but with the central axis now being on a diameter of the sphere, once the base touches the sphere, the entire circle at the base of the cone would be contained in the sphere. So now a plane passing through that central axis would show an isoceles triangle touching a circle in three points.

If we draw coordinate axes with the center of the circle as the origin and with the axis of the cone as the y-axis, then the circle is the set of points (x, y) having $x^2 + y^2 = 1$. The apex of the cone is the point $(0, 1)$ and the other two vertices are the points $(\pm a, b)$ for some $a > 0$ and b having $a^2 + b^2 = 1$. The height h of the cone would be $1 - b$ and the radius of its circular base would be a, so the volume would be $(1/3)hA = (1/3)(1-b)(\pi a^2)$. Since $a^2 + b^2 = 1$ this may be written $(\pi/3)(1 - b)(1 - b^2)$.

So our goal is to maximize the value of this function of b, on the interval $[-1, 1]$. We may use two important theorems: (1) A continuous function on a closed and bounded interval must attain a maximum somewhere, and (2) if the function attains a maximum at a point b then the first derivative vanishes at b (or fails to exist at b) unless b is an endpoint. Well, the derivative $(\pi/3)(-1 - 2b + 3b^2)$ vanishes iff $b = (2 \pm 4)/6$ i.e. iff $b = 1$ or $b = -1/3$. With the endpoints there are then only three candidate values of b to maximize this function: $b = -1, b = -1/3, b = +1$, and they give values of $0, 32\pi/81, 0$ respectively. Clearly the middle option is the best of these three.

So the largest possible cone has a height of $\frac{4}{3}$, a radius of $\frac{2\sqrt{2}}{3}$, and a volume of $\frac{32}{81}\pi$.

2. (i) Find the distance from the point $(3, 6, 5)$ to the plane $x + 2y + 3z = 2$.

(ii) Find the distance between the parallel planes $x + 2y + 3z = 2$ and $x + 2y + 3z = 0$.

In working parts (i) and (ii) do not use the general formula for the distance from a point to a plane for the distance between two parallel planes unless you prove it.

ANSWER: The distance from the point $P = (3, 6, 5)$ to another point (x, y, z) is of course $\sqrt{(x-3)^2 + (y-6)^2 + (z-5)^2}$. If this latter point is known to lie on the given plane then $x = 2 - 2y - 3z$ so the distance may be rewritten as $\sqrt{(1 + 2y + 3z)^2 + (y-6)^2 + (z-5)^2}$. When we speak of the distance from P to a plane, we mean the minimum of all possible distances to the points in the plane, so we simply choose y and z so as to make this distance minimal. It is sufficient to make the *square* of that distance minimal (since squaring is an increasing function), and that will happen when the two partial derivatives vanish; that is, the closest point of the plane to P has

$$2(1 + 2y + 3z)^1(2) + 2(y-6)^1 = 0 \qquad \text{and} \qquad 2(1 + 2y + 3z)^1(3) + 2(z-5)^1 = 0$$

These are two linear equations $5y + 6z = 4, 6y + 10z = 2$ whose unique solution is $y = 2, z = -1$. The point Q on the plane with these coordinates also has $x = 2 - 2y - 3z = 1$. The distance from $P = (3, 6, 5)$ to $Q = (1, 2, -1)$ is $2\sqrt{14}$.

In exactly the same way we can determine the distance from any point $P = (a, b, c)$ to this plane; the point on the plane that is closest to P turns out to be

$$\frac{1}{14}(2 + 13a - 3c - 2b, 4 - 2a - 6c + 10b, 6 - 3a + 5c - 6b)$$

whose distance from P is

$$\frac{|a + 2b + 3c - 2|}{\sqrt{14}}$$

(NB: This is the "formula" to which the problem refers: the distance from (a, b, c) to the plane $Ax + By + Cz = D$ is $\frac{|Aa+Bb+Cc-D|}{\sqrt{A^2+B^2+C^2}}$.)

So for example the distance from the point $(-27, 6, 5)$ to the given plane would be $|(-27) + 12 + 15 - 2|/\sqrt{14} = 2/\sqrt{14} = \sqrt{14}/7$. I mention this point because it lies in the other plane described in (ii). Of course when two planes are parallel, *every* point in the one plane is the same distance from the other plane, so $2/\sqrt{14}$ is then the distance between the planes.

An alternative solution simply computes the length of a perpendicular vector $\lambda\langle 1, 2, 3\rangle$ which must be added to any point in the one plane to obtain a point in the other plane; for example starting from the origin we want $\lambda + 2(2\lambda) + 3(3\lambda) = 2$, which requires $\lambda = 1/7$, meaning that the origin (which is in one plane) is nearest to $(1/7, 2/7, 3/7)$ in the other plane; the distance between these is $\sqrt{14}/7$.

3. Compute the sum $\sum_{n=0}^{\infty}(3+(-1)^n)^{-n}$.

(Hint: Write the first several terms of the series.)

ANSWER: Taking the hint, we write the series as

$$\frac{1}{4^0}+\frac{1}{2^1}+\frac{1}{4^2}+\frac{1}{2^3}+\frac{1}{4^4}+\frac{1}{2^5}+\ldots$$

This series is absolutely convergent (each term is positive and bounded by the terms of the convergent geometric series $\sum 1/2^n$) so we may rearrange the terms without affecting the sum. The terms with powers of 4 in them are

$$\frac{1}{4^0}+\frac{1}{4^2}+\frac{1}{4^4}+\ldots=\frac{1}{16^0}+\frac{1}{16^1}+\frac{1}{16^2}+\ldots$$

which is a geometric series with sum $1/(1-(1/16)) = 16/15$. The other terms in the original series are

$$\frac{1}{2^1}+\frac{1}{2^3}+\frac{1}{2^5}+\ldots$$

which is a geometric series too, and sums to $(1/2)/(1-(1/4)) = 2/3$.

Thus the original series sums to $16/15 + 2/3 = 26/15$.

4. Find the equation of each line which passes through the origin and is tangent to the curve $y = x^4 + x^3 - x^2 + 2x$ at some point.

ANSWER: The point on the curve that has x coordinate equal to some number a obviously has a y coordinate of $b = a^4 + a^3 - a^2 + 2a$. The slope of the tangent line at this point is $m = 4a^3 + 3a^2 - 2a + 2$. On the other hand the slope of the line that connects (a, b) to the origin is $m' = b/a = a^3 + a^2 - a + 2$. These lines will be equal iff they have the same slope; but $m = m'$ iff $0 = 3a^3 + 2a^2 - a = a(3a-1)(a+1)$. So the points in question are those with $a = 0, 1/3$, or -1. In each case the line is $y = m'x$, so we find the lines to be these three:

$$y = 2x, \qquad y = \frac{49}{27}x, \qquad y = 3x$$

5. Compute $\displaystyle\int_{-1}^{0} \frac{1}{(x+2)^3\sqrt{x^2+4x+3}}\,dx.$

ANSWER: Note that this is am improper integral, the integrand being undefined at $x=-1$. So we will need to find an antiderivative to evaluate $\int_a^0 f(x)\,dx$ for $a \to (-1)^+$.

A sequence of small substitutions will make the integrand better. First let $x = u-2$ so the integral is

$$\int \frac{1}{u^3\sqrt{u^2-1}}du$$

Then write $u = \sec(\theta)$, where (since $u \in [1,2]$) we may assume $\theta > 0$ and thus $\sqrt{u^2-1} = \tan(\theta)$ (rather than $-\tan(\theta)$). Then $du = \sec(\theta)\tan(\theta)\,d\theta$ and the integral becomes

$$\int \frac{1}{\sec^2(\theta)}\,d\theta = \int \cos^2(\theta)\,d\theta$$

This is a familiar integral: we may use a trigonometric identity to write $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$ and then integrate to get an antiderivative $\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) = u\frac{1}{2}\theta + \frac{1}{2}\sin(\theta)\cos(\theta)$. In terms of the previous variables this is

$$\frac{1}{2}\arccos\left(\frac{1}{u}\right) + \frac{1}{2}\sqrt{1-\left(\frac{1}{u}\right)^2}\left(\frac{1}{u}\right)$$

and then

$$\frac{1}{2}\arccos\left(\frac{1}{x+2}\right) + \frac{\sqrt{x^2+4x+3}}{2(x+2)^2}$$

At $x=0$ this has the value $\pi/6 + \sqrt{3}/8$, and as x approaches -1 from above, the value approaches 0. So the value of the improper integral is exactly

$$\frac{\pi}{6} + \frac{\sqrt{3}}{8}$$

Alternatively, one might use the substitution $u = \sqrt{v+1}$ and then $v = w^2$ to get a rational integrand, which is a standard Calculus-II topic.