Here are some possible answers to the questions on the
Albert A. Bennett Calculus Prize Exam — May 7 2011

Note: It is quite possible that the same answers can be obtained by other methods. Indeed, creative means to derive correct answers are valued in mathematics generally and in this contest in particular.

1. Determine whether these series converge or diverge. (Be sure to justify your answer.)

$$(a)\ \sum_{n=1}^{\infty} \frac{1}{(3n-2)^{n+(1/2)}} \qquad\qquad (b)\ \sum_{n=1}^{\infty} \frac{(-1)^{n(n-1)/2}}{n} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots$$

Solution: Series (a) clearly converges: for $n > 1$ we have $3n-2 > 3$ and $n + (1/2) > n$ so the denominator is larger than 3^n. Thus our series is dominated by $\sum_{n=1}^{\infty} 3^{-n}$, a geometric series with common ratio $1/3 < 1$. Since this series converges, so does ours.

Series (b) also converges. The pattern of signs $(+,-,-,+)$ repeats every four terms (because $(n+4)(n+3)/2 - n(n-1)/2$ is even), and so a typical sum of four terms is

$$\frac{1}{4n-3} - \frac{1}{4n-2} - \frac{1}{4n-1} + \frac{1}{4n}$$

which simplifies to $a_n = (8n-3)/(4n(4n-1)(4n-2)(4n-3))$; use the Limit Comparison Test to compare to the p-series $\sum n^{-3}$, which converges, to see that our series converges. (It is also sufficient to take the terms simply in pairs; the resulting series converges absolutely.)

Remark: The Alternating Series Test does *not* apply. (Why?)

2. Compute the following limit, or show that it does not exist:

$$\lim_{x\to 0} \frac{x^3 \sin(\frac{1}{x})}{\ln(1+x^2)}$$

Solution: Note that $|\sin(u)| \le 1$ for all u, so the numerator is smaller than x^3 in magnitude; on the other hand, the denominator is approximately x^2. (Its Taylor series is an alternating series showing that it's between x^2 and $x^2 - x^4/2$.) Thus the value of the function is bounded in magnitude by $x^3/(x^2 - x^4/2) = x/(1 - x^2/2)$, which tends to zero as $x \to 0$; so the original limit is zero.

Remark: An attempt to use L'Hopital's Rule leads to the limit

$$\lim_{x\to 0} \frac{\left(3x\ \sin(\frac{1}{x}) - \cos(\frac{1}{x})\right)\left(1+x^2\right)}{2}$$

which does not exist. But L'Hopital's Theorem only says that *if* this second limit exists, *then* it equals the original limit; so that Theorem gives no information here. ("LHR does not apply.")

3. Compute the first three terms $a_0 + a_1 x + a_2 x^2$ of the Maclaurin series (i.e. the Taylor series at 0) for

$$f(x) = \frac{5x-7}{(x-1)(x-2)}$$

Solution: Compute the partial fractions decomposition of f: it's

$$f(x) = \frac{2}{x-1} + \frac{3}{x-2} = \frac{-2}{1-x} + \frac{-3/2}{1-x/2}$$

So we can write the entire function as a sum of two geometric series:

$$f(x) = (-2)\sum x^n + (-3/2)\sum (x^2/2^n) = -7/2 - (11/4)x - (19/8)x^2 - \dots$$

4. Find a point which is equidistant from all four planes

$$x = 0 \qquad y = 0 \qquad z = 0 \qquad 2x + 3y + 6z = 36.$$

Solution: The first three conditions place the point on the line $x = y = z$. Now, the distance from (x, y, z) to the last plane is $|2x + 3y + 6z - 36|/\sqrt{2^2 + 3^2 + 6^2}$, so a point (t, t, t) is of distance t to each of the first 3 planes and of distance $|11t - 36|/7$ from the last, and so we need $11t - 36 = \pm 7t$; we need $t = 2$ or $t = 9$, so the possible solutions are the points (2,2,2) and (9,9,9).

CORRECTION: The above is incomplete. The distance from (x, y, z) to the plane $x = 0$ is $|x|$, not x, and similarly for the other two coordinate planes. Thus the fact that the point should be equidistant to the first three planes only forces the point to lie on ONE of the lines $x = y = z$, $x = y = -z$, $x = -y = z$, or $-x = y = z$. Each of these lines contains two solution points in the same manner as above: we would need, respectively, $|11t - 36|/7 = |t|$, $|-t - 36|/7 = |t|$, $|5t - 36|/7 = |t|$, or $|7t - 36|/7 = |t|$, leading to the points (2,2,2) and (9,9,9); (6,6,-6) and (-9/2,-9/2,9/2); (3,-3,3) and (18,-18,18); and (-18/7, 18/7, 18/7) respectively: seven solutions.

5. Find all the critical points of the function below, and state whether they are local minima, local maxima, or saddle points:

$$f(x, y) = 1 - (x^2 - 1)^2 - (x^2 y - x - 1)^2.$$

Solution: Since squares are always non-negative, we have $f \le 1$ everywhere and $f = 1$ precisely when $x^2 - 1 = x^2 y - x - 1 = 0$; this happens when $x = \pm 1$ and then (respectively) $y = 2$ or $y = 0$. So these two points (1,2) and (0,2) give local maxima. (They're actually global maxima.)

At any critical point (x, y) we must have $0 = f_x(x, y) = f_y(x, y)$ i.e.

$$-2(x^2 - 1)(2x) - 2(x^2 y - x - 1)(2xy - 1) = 0, \qquad -2(x^2 y - x - 1)x^2 = 0$$

The second equation shows that either $x = 0$ or $x^2 y - x - 1 = 0$. The first possibility is inconsistent with the first equation, however, and if instead $x^2 y - x - 1 = 0$ then the first equation requires $x = 0$ (which we have already ruled out) or $x = \pm 1$ (which we have discussed in the previous paragraph). So there are no other critical points.

Remark: This example describes a surface with "two mountains but no valley"—there's not even a mountain pass between the two mountains!