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The Bennett Linear Algebra Competition questions will revolve around topics that should be treated in both M340L and M341. These include:

Linear Equations in Linear Algebra,
Matrix Algebra (including products and inverses),
Determinants,
Vector Spaces,
Linear Transformations,
Eigenvalues and Eigenvectors,
Orthogonality and Inner Products

Students are also expected to be comfortable with topics from Calculus-1 (a prerequisite for the Linear Algebra courses) and of course the pre-calculus material which is *its* pre-requisite.

The questions will not require that students present formal proofs, but some of the questions will require an explanation of why such-and-such a conclusion is justified. (It is unlikely that a problem on this competition can be solved simply by some routine row-reduction, for example. You will have to compute something in a clever way and then explain why that method is appropriate.) Consider an example:

Sample problem: Is the $n \times n$ matrix M for which $M_{ij} = i + j$ invertible?

Sample solution: Observe that when $n = 1$ and $n = 2$ the matrix *is* invertible: the matrices and their inverses are

$$(2)^{-1} = (\frac{1}{2}) \qquad \text{and} \qquad \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$$

But for larger values of n we find that $\det(M) = 0$, so that the matrix is not invertible. This follows since the first three rows are linearly dependent; in fact, the second row is the average of the first and third rows (the j entries in the three rows are $j+1, j+2$, and $j+3$ and for every j we check that indeed $j+2$ is the average $((j+1)+(j+3)/2)$.

Comment: it would not be sufficient to handle to case $n = 3$ alone, but if you compute the determinant for $n = 3$ you can simply say the remaining cases are similar *if you use an argument involving row operations as above* which does indeed apply to matrices of all sizes. If instead you compute $\det(M) = 0$ when $n = 3$ by the technique of multiplying entries along various diagonals, then you really haven't explained what will happen for larger values of n.

Notation: depending on your instructor and his or her choice of textbook, you may have learned different words and notations for common linear-algebraic ideas. You may use the notation of your choice in your answers but here we list a few synonyms to make sure you understand the questions in this contest. (If you are in doubt, ask the exam supervisor.)

Vectors may be written with or without special notation like overhead arrows or the use of boldface. In particular the symbol "0" may refer either to a number, or to a vector in $\mathbf{R}^n$, or to the identity element in an abstract vector space, or to the zero matrix of any size (square or not)!

You may treat $\mathbf{R}^n$ to mean either the set of row vectors of length n, or the set of column vectors of length n, or the set of points Euclidean n-dimensional space, although in some contexts involving matrix multiplication or geometry only one of these may make sense.

The transpose of a matrix A might be denoted either A^t or A^T or A'

The trace of a square matrix (the sum of its diagonal entries) may be denoted $\text{tr}(A)$ or $\text{Tr}(A)$ or $\text{trace}(A)$.

The characteristic polynomial of a square matrix A might be $\det(xI-A)$ or $\det(A-xI)$.

Functions defined on a vector space and giving outputs in another vectors space may be called *linear maps* or *linear transformations* or *linear functions* but all these terms refer to functions that meet the same two axioms of linearity. For example, you should be comfortable with the idea that each rotation around the origin in the plane is a linear map $L : \mathbf{R}^2 \to \mathbf{R}^2$.

If $F : V \to W$ is a linear map, the set of vectors v in V for which $F(v) = 0$ is called the *null space* of F or the *kernel* of F. Likewise if A is a matrix: the vectors v of suitable size for which $Av = 0$ may be called either the null space or the kernel of A. The notations $\ker(F)$ and $\text{null}(F)$ are synonyms, and likewise $\ker(A)$=$\text{null}(A)$.

Similarly the *range* and the *image* of a function should be thought of as synonyms. (However, if a function is presented as a map $F : V \to W$, the proper name for W is to call it the *codomain* of F; some people refer to W as the range of F but that is generally incorrect. For example if $F : \mathbf{R}^2 \to \mathbf{R}^3$ is the zero map, that is, F amounts to multiplication by the 3×2 zero matrix, then the domain of F is $\mathbf{R}^2$, the codomain is $\mathbf{R}^3$, and the image is just the zero vector in $\mathbf{R}^3$; for clarity you should probably avoid the use of the word "range".)

To say a matrix is *diagonalizable* is to say that it is *conjugate to* a diagonal matrix or *similar to* a diagonal matrix. Both these words are synonyms and may be expressed as follows: A is conjugate to B if their exists an invertible matrix P for which $A = PBP^{-1}$, i.e. for which $AP = PB$. It is equivalent to saying there is an invertible matrix Q for which $A = Q^{-1}BQ$, i.e. for which $QA = BQ$. It is also equivalent to saying there are two matrices P and Q which are inverses to each other for which $A = PBQ$.

For the purposes of this exam, an *inner product* will be taken to be synonymous with the *dot product* in $\mathbf{R}^n$, although you may write the inner product of two vectors as either $v \cdot w$ or $\langle v, w \rangle$ following the notation used in your class. Note that *perpendicular*, *orthogonal*, and *normal* are used interchangeably when the inner products between two vectors (or two subspaces) are zero.