THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

The Preliminary Examination in Probability
Part I

Fri, Jan 12, 2024

Problem 1. Let $T : [0, 1] \to [0, 1]$ be given by

$$T(0) = 0 \text{ and } T(x) = \{1/x\} \text{ for } x \in (0, 1],$$

where $\{a\} := \sup\{a - m : m \in \mathbb{Z} \text{ and } m \leq a\} = a = \lfloor a \rfloor$ is the fractional part of a. Show that $T_*\mu = \mu$, where $T_*\mu$ is the pushforward of μ via T and

$$\mu(B) = \int_B \frac{1}{1+x}\, dx \text{ for } B \in \mathcal{B}([0, 1]).$$

(*Note:* T is (one of several things) known as the **Gauss map**.)

Problem 2. Let μ be a probability measure on $\mathbb{R}$, and let φ be its characteristic function. Show that μ is diffuse (has no atoms) if

$$\lim_{t\to\infty} |\varphi(t)| = \lim_{t\to-\infty} |\varphi(t)| = 0.$$

(*Hint:* For $a \in \mathbb{R}$, compute $\lim_{T\to\infty} \int_{-T}^{T} e^{-ita}\varphi(t)\, dt$.)

Problem 3. Given $X \in \mathbb{L}^2(\mathcal{F})$ and two sub-σ-algebras $\mathcal{G}, \mathcal{H}$ of $\mathcal{F}$ such that $\mathcal{G} \subseteq \mathcal{H}$, show that

$$\mathbb{E}[\mathrm{Var}[X \mid \mathcal{G}]] \geq \mathbb{E}[\mathrm{Var}[X \mid \mathcal{H}]],$$

where $\mathrm{Var}[X \mid \mathcal{K}] := \mathbb{E}\big[(X - \mathbb{E}[X \mid \mathcal{K}])^2 \,\big|\, \mathcal{K}\big]$ for $\mathcal{K} \subseteq \mathcal{F}$. When does the equality hold?

(*Note:* $\mathrm{Var}[X \mid \mathcal{K}]$ is called the **conditional variance** of X given $\mathcal{K}$.)