## ALGEBRA PRELIMINARY EXAM: PART II

## Problem 1

Let p be a prime,  $\mathbb{F}_p$  be the field with p elements, and F a finite field of characteristic p.

- a) Prove that F has  $p^n$  elements for some positive integer n.
- b) Prove that  $\prod_{\alpha \in F^{\times}} \alpha = -1$  where  $F^{\times} = F \setminus \{0\}$ .
- c) For any  $a \in \mathbb{F}_p$  we consider  $f_a(x) := x^p x + a \in \mathbb{F}_p[x]$ . Prove that  $f_a$  is separable and irreducible over  $\mathbb{F}_p$  for every  $a \in \mathbb{F}_p \setminus 0$ .

## Problem 2

Consider the polynomial  $f(x) = x^7 - 3 \in \mathbb{Q}[x]$ . Let  $E/\mathbb{Q}$  denote the splitting field of f(x).

- a) Determine the Galois group of  $E/\mathbb{Q}$  as an abstract group and by describing its elements in terms of their action on the generators of E.
- b) Determine a primitive generator for  $E/\mathbb{Q}$ .
- c) Identify all subfields  $E/\mathbb{Q}$  which are Galois over  $\mathbb{Q}$ .

## Problem 3

Consider polynomial  $f(x) = x^4 - 7x + 7 \in \mathbb{Q}[x]$ .

- a) Determine the Galois group of f(x) over  $\mathbb{Q}$ .
- b) Does there exist  $n \in \mathbb{N}$  such that f(x) has a root in  $\mathbb{Q}(\zeta_n)$ ?

You may use without proof the fact that

$$Disc(f) = (-1)^{(4\cdot3/2)}4^47^4 + (-1)^{(3\cdot2/2)}3^3(-7)^4 = 7^3(4^4 - 3^3 \cdot 7) = 7^3 \cdot 67.$$

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