PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part I

Wednesday, August 21, 2024, 11:30am-1:30pm

Work all 3 of the following 3 problems.

1. Let X and Y be Banach spaces and D a linear subspace of X.

(**a**) State what it means for an operator $T : D \to Y$ to be closed.

(**b**) State the Closed Graph Theorem.

(**c**) Let $T : D \to Y$ be a closed linear operator. Show that T is bounded on D if and only if D is a closed subspace of X.

2. Let X be a Banach space, Y a normed linear space, and $T_n \in B(X, Y)$ for $n = 1, 2, \ldots$. Suppose that $\{T_n x\}_{n=1}^{\infty}$ is Cauchy in Y for every $x \in X$.

(**a**) State the Uniform Boundedness Theorem.

(**b**) Show that $\{\|T_n\|\}_{n=1}^{\infty}$ is bounded.

(**c**) If Y is a Banach space and T is defined by $T_n x \to Tx$ for $x \in X$, show that T is a well-defined operator and that $T \in B(X, Y)$.

3. Let H be a separable Hilbert space with maximal orthonormal basis $\{u_k\}_{k=1}^{\infty}$, let $H_n = \text{span}\{u_1, \ldots, u_n\}$, and let P_n denote the orthogonal projection of H onto H_n. Suppose that $A : H \to H$ is bounded and linear and $f \in H$. If for all n

$$P_n A x_n = P_n f$$

has a solution $x_n \in H_n$ such that

$$\|x_n\| \leq \alpha \|P_n f\|,$$

where $\alpha > 0$ is independent of n.

(**a**) Recall that $\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, u_k \rangle|^2$. From this fact, prove that $P_n x \to x$ as $n \to \infty$.

(**b**) Show that there is at least one solution to $Ax = f$.