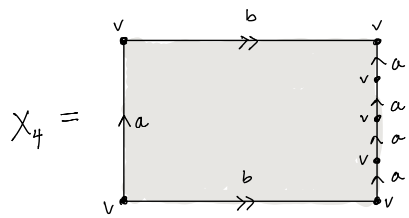
Preliminary Examination: Algebraic topology. August 20, 2024

Instructions: Answer all three questions. Question 1 is worth 4 points. Questions 2 is worth 2 points. Question 3 is worth 2 points.

Time limit: 2 hours.

1. For each $d \in \mathbb{Z}$, consider the map $f_d : S^1 \to S^1$ that wraps the circle around itself d times: $f_d(e^{i\theta}) = e^{di\theta}$. Let $X_d = S^1 \times I/\sim$ be the mapping torus of f_d, defined by the relation $(x, 0) \sim (f_d(x), 1)$. The figure suggests a cell structure on X_d.

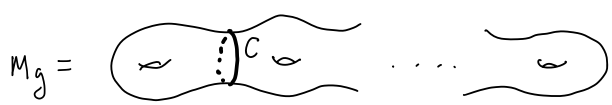


(a): For each $d \neq 1$, compute the homology groups $H_*(X_d)$.

(b): For which values of d does X_d retract onto the circle $S^1 \times \{1/2\}$?

(c): For which values of d is X_d homotopy equivalent to a surface?

(d): Now let the space $Y = Y_{d,g}$ be obtained from X_d and a closed orientable surface M_g of genus $g \geq 2$ by identifying $S^1 \times \{1/2\} \subset X_d$ with the curve $C \subset M_g$ shown in the figure. Use a Mayer-Vietoris sequence to compute the homology groups $H_*(Y)$.



2. Let $X = S^1 \vee S^1$ be the wedge of two circles, and let $x \in X$ be the point where the two circles meet (the "wedge" point). Then $\pi_1(X, x) \cong F_2 = \langle a, b \rangle$ is the free group on two generators.

(a): Describe all path-connected covering spaces of X of degree two. [*Note:* it is acceptable to describe a covering space by drawing a well-labeled picture.]

(b): Using your answer to (a), list all subgroups of index two in F_2 and give a geometric proof that every such subgroup is normal. [*Note:* It is a general fact that any index two subgroup in an arbitrary group G is normal, but do not use this general fact. The point is to prove this fact using covering space theory in the case $G = F_2$.]

3. Let G be a group.

(a): Briefly explain the general construction of a basepointed 2-dimensional cell complex (X_G, x_0), so that $\pi_1(X, x_0) = G$. Demonstrate the construction in the case $G = \langle a, b, c : a^2b, c^3 \rangle$.

(b): Let (Y, y_0) be a path connected topological space with basepoint. Let $h : G \to \pi_1(Y, y_0)$ be a group homomorphism. Briefly describe a general construction of a continuous map $f : (X_G, x_0) \to (Y, y_0)$ such that $f_* = h$, where here f_* denotes the induced map on fundamental group.