Differential Topology Prelim Exam, August 2024

High-quality answers to some questions are preferred to lower-quality answers to more questions. Point allocations are shown on the right.

(1) Let p be a monic polynomial whose real roots are distinct. Let
$$S(p) = \{(x,y,z) \in \mathbb{R}^3 : p(x) + y^2 + z^2 = 0\}.$$

(a) Prove that $S(p)$ is a submanifold of $\mathbb{R}^3$.

[3 points]

(b) Assuming p has even degree, identify $S(p)$ up to diffeomorphism with a familiar manifold (depending on p).

[3]

(c) Let q be another monic polynomial whose real roots are distinct. Characterize when $S(p)$ is transverse to $S(q)$.

[4]

(2) (a) Consider the 1-form $d\theta$ on $S^1 \subset \mathbb{R}^2$ (here θ is the polar angle). Prove that there is no C^∞ function $f\colon S^1 \to \mathbb{R}$ with $d\theta = df$.

[3]

(b) Define $\rho\colon \mathbb{R}^2 \setminus \{0\} \to S^1$, $\rho(v) = v/|v|$. Compute $\rho^*(d\theta)$ in terms of the (x,y)-coordinates of $\mathbb{R}^2$.

[3]

(c) Define the 1-form $\alpha = y\,dx - x\,dy$ on $\mathbb{R}^2 \setminus \{0\}$. Prove that there do not exist a C^∞ map $g\colon \mathbb{R}^2 \setminus \{0\} \to S^1$ and a 1-form β on S^1 such that $\alpha = g^*\beta$.

[4]

(3) This question is about the Grassmannian $Gr_k(\mathbb{R}^n)$ of k-dimensional vector subspaces of $\mathbb{R}^n$. $Gr_k(\mathbb{R}^n)$ is covered by the subsets $\{G(U) : U \in Gr_k(V)\}$, where $G(U)$ consists of the subspaces W such that the orthogonal projection $p_U\colon \mathbb{R}^n \to U$ restricts to an isomorphism $p_U|_W\colon W \to U$. It is proposed to construct a C^∞ atlas on $Gr_k(\mathbb{R}^n)$ in which G_U is the domain of a chart.

(a) By thinking of $W \in G(U)$ as the graph of a linear map, exhibit a bijection $\phi_U\colon G(U) \to \mathsf{Hom}(U, U^\perp)$ with the vector space of linear maps.

[2]

(b) Suppose that $W \in Gr_k(\mathbb{R}^n)$ is the graph of $\alpha\colon U \to U^\perp$ and of $\beta\colon V \to V^\perp$. Explain why, if $u + \alpha u = v + \beta v \in W$, we have
$$v = p_V(u + \alpha u),$$
and that this formula defines an isomorphism $\mu_\alpha = p_V \circ (I + \alpha)\colon U \to V$. Show that
$$\beta v = (I + \alpha)\mu_\alpha^{-1} v - v.$$

[3]

(c) Explain concisely why $\{(G(U), \phi_U) : U \in Gr_k(V)\}$ is a C^∞ atlas for the Grassmannian. [You are *not* asked to prove that the topology is Hausdorff and second countable.]

[5]