Real Analysis Prelim Spring 2023 - January 10, 2023

Solve all three problems:

1. Consider the Hardy-Littlewood maximal function (in balls)

$$Mf(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f|, \qquad \text{for } f(x) \text{ a continuous, compactly supported function in } \mathbb{R}^n,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x.

Prove that $Mf(x)$ belongs to $L^1_{\text{weak}}(\mathbb{R}^n)$.

2. Let $f \in L^1(\mathbb{R})$ and φ_ε be a mollifier, that is $\varphi_\varepsilon = \varepsilon^{-1}\varphi(x/\varepsilon)$, $\varphi : \mathbb{R} \to \mathbb{R}$ is a function satisfying $\varphi \geq 0$, the support of φ is compact and $\int \varphi = 1$.

Let $f_\varepsilon = f * \varphi_\varepsilon$ be the convolution. Show that

$$\int_{\mathbb{R}} \liminf_{\varepsilon \to 0} |f_\varepsilon| \leq \int_{\mathbb{R}} |f|.$$

3. Let $1 \leq p \leq \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$. Show that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^{\overset{q}{\cancel{p}}}(\mathbb{R}^n)$, then

(a) the convolution $f * g$ is bounded and continuous on $\mathbb{R}^n$.

(b) In addition, prove that if either $f \in (L^p \cap C^m)(\mathbb{R}^n)$, or $g \in (L^q \cap C^m)(\mathbb{R}^n)$, then $f * g \in C^m(\mathbb{R}^n)$.