PRELIMINARY EXAMINATION IN ANALYSIS

PART I

AUGUST 2023

Please provide as-complete-as-possible proofs for all of the following 4 problems.

- (1) Let $1 and <math>f_n \in L^p([0,1],\lambda)$ (n = 1,2,...) and $f \in L^p([0,1],\lambda)$ (where λ is Lebesgue measure). Let M > 0 be a constant and suppose
 - (a) f_n converges to f pointwise a.e.;
 - (b) $||f_n||_p \leq M$ for all n.

Prove that for every $g \in L^q([0,1],\lambda)$ $(\frac{1}{p} + \frac{1}{q} = 1)$,

$$\lim_{n \to \infty} \int f_n g \, d\lambda = \int f g \, d\lambda.$$

Warning: do not assume f_n converges to f in $L^p([0,1],\lambda)$.

(2) Let ν be a finite Borel measure on [0,1]. Define $f:[0,1]\to\mathbb{R}$ by

$$f(x) = \nu([0, x)).$$

Prove ν is absolutely continuous to Lebesgue measure if and only if f is absolutely continuous.

- (3) Given a function $\phi : \mathbb{R} \to [0, \infty)$ and $\epsilon > 0$, define $\phi_{\epsilon} : \mathbb{R} \to \mathbb{R}$ by $\phi_{\epsilon}(x) = \epsilon^{-1}\phi(x/\epsilon)$. Let $f \in L^{\infty}(\mathbb{R}, \lambda)$ (where λ is Lebesgue measure).
 - (a) Fix 0 < a < 1 and let $\psi = (1/2a)1_{[-a,a]}$ (where $1_{[-a,a]}$ is the characteristic function of [-a,a]). Explain why the convolution $f * \psi_{\epsilon}$ converges to f pointwise a.e. as $\epsilon \searrow 0$.
 - (b) Suppose $\phi : \mathbb{R} \to [0, \infty)$ be a continuous function which is even (so $\phi(x) = \phi(-x)$ for all x), has $\phi(x) = 0$ for all $|x| \ge 1$ and has $||\phi||_1 = 1$. Prove $f * \phi_{\epsilon} \to f$ pointwise a.e. as $\epsilon \searrow 0$.
- (4) Let μ_n be a sequence of Borel probability measures on [0,1] which converges to a probability measure μ in the weak* topology. This means: for every continuous function f on [0,1], $\int f d\mu = \lim_{n\to\infty} \int f d\mu_n$.

Let $C \subset [0,1]$ be closed. Prove $\limsup_{n\to\infty} \mu_n(C) \leq \mu(C)$.