ALGEBRA PRELIMINARY EXAM: PART I

Problem 1

a) Let L be a field, $V := L^n$ the L-vector space of column vectors with n entries, and $R := L[T]$ the ring of polynomials with coefficients in L.
   i) Explain why an R-module structure on V, restricting to its natural L-module structure, determines and is determined by an $n \times n$ matrix over L.
   ii) For an $n \times n$ matrix A, let V_A be the corresponding R-module. Show that V_A,V_B are isomorphic if and only if the matrices A and B are conjugate.

b) Let $L \subset K$ be a field extension. Let A, B be $n \times n$ matrices over L which are conjugate over K. Are they necessarily conjugate over L? If so, prove it; if not, give a counter-example.

Problem 2

a) Prove that a finite group is not the union of conjugates of any proper subgroup.

b) Show by way of example that this fails in general for infinite groups. Equivalently: give an example of a proper subgroup H of a group G such that every element of G is conjugate to an element of H.

Problem 3

Let p be the smallest prime dividing the order of a finite group G, and $H \subset G$ a subgroup of index p. Show that H is normal.

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