

**Applied Mathematics Preliminary Exam, Part A**  
August 18, 2023

---

*Work 4 of the following 5 problems.*

- 1.** If  $f$  is a map from a Banach space  $X$  to a Banach space  $Y$ , such that  $\psi \circ f$  belongs to the dual of  $X$  for every  $\psi$  in the dual of  $Y$ , show that  $f$  is linear and continuous.
  
- 2.** Let  $P$  be a linear operator on a Banach space, satisfying  $P^2 = P$ . Show that the operator  $P$  is continuous if and only if its null space and range are both closed.
  
- 3.** Show that a compact linear operator maps weakly convergent sequences to strongly convergent sequences.
  
- 4.** If  $U$  is an unitary operator on a Hilbert space, show that  $n^{-1}[I + U + U^2 + \dots + U^{n-1}]$  converges strongly to an orthogonal projection, as  $n \rightarrow \infty$ .
  
- 5.** For  $m = 1, 2, 3, \dots$ , define  $H_m$  to be the regular distribution on  $\mathbb{R}$  associated with the function  $h_m(x) = m^2 \sin(mx)$ . Show that  $H_m \rightarrow 0$  in  $\mathcal{D}'(\mathbb{R})$  as  $m \rightarrow \infty$ .