## PRELIMINARY EXAMINATION: APPLIED MATHEMATICS—Part II

August 18, 2023, 3:00-5:00pm

Work all 3 of the following 3 problems.

- **1.** If  $d \ge 1$  and 0 < k < d are integers, the restriction map  $R : C^0(\mathbb{R}^d) \to C^0(\mathbb{R}^{d-k})$  is defined for  $y \in \mathbb{R}^{d-k}$  by  $R\varphi(y) = \varphi(y,0)$ .
  - (a) Define the Sobolev space  $H^s(\mathbb{R}^d)$  for any  $s \geq 0$  in terms of the Fourier Transform.
  - (b) Show that for  $\varphi \in C^0(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ ,

$$\widehat{R\varphi}(\eta) = (2\pi)^{-k/2} \int_{\mathbb{R}^k} \widehat{\varphi}(\eta, \zeta) \, d\zeta.$$

- (c) Show that R extends to a bounded linear map from  $H^s(\mathbb{R}^d)$  into  $H^{s-k/2}(\mathbb{R}^{d-k})$ , provided that s > k/2. [You do not need to show that R maps onto, although it does.]
- **2.** For a given bounded domain  $\Omega \subset \mathbb{R}^d$  with a smooth boundary and constant  $K \geq 0$ , consider the problem of finding u solving

$$\Delta u - K u = f$$
 in  $\Omega$ ,  
 $\nabla u(x) \cdot \nu = g(x)$  for  $x \in \partial \Omega$ .

for a given  $f \in H^s(\Omega)$  and g(x).

- (a) To what Sobolev space must g(x) belong, in order that the Neumann boundary value problem has a solution  $u \in H^{s+2}(\Omega)$ ?
- (b) Give estimates for the solution u in the  $H^{s+2}(\Omega)$ -norm showing the dependence with respect to the data f, g, and K in terms of their Sobolev norms.
- (c) Do f and g need to satisfy a compatibility condition? If so, give the condition. The answer depends on K.
- (d) For what values of s will u be continuous?
- **3.** Show that the Fredholm integral equation

$$f(x) = \phi(x) + \lambda \int_{a}^{b} K(x, y) f(y) dy$$

has a unique solution  $f \in C([a,b])$  provided that  $\lambda$  is sufficiently small, wherein  $\phi \in C([a,b])$  and  $K \in C([a,b] \times [a,b])$