Differential Topology Prelim Exam August 15, 2023, 15:00-17:00

Solve all three problems.

Problem 1. True or false with brief explanations, including counterexamples when

- (a) Suppose M is a manifold and $f: M \to \mathbb{R}$ is smooth. Then f is a Morse function if and only if every one of its critical points is isolated.
 - (b) Every smooth map $\mathbb{C}P^4 \to S^8$ is nullhomotopic. (c) Every smooth map $\mathbb{C}P^4 \to S^9$ is nullhomotopic.

Problem 2. Let X be the subset $x^n - y^2 = 0$ of \mathbb{C}^2 , where n > 2 is even. Prove that X is the union of two smooth submanifolds of \mathbb{C}^2 , that meet each other nontransversely. Describe the natural orientations on these submanifolds, and on \mathbb{C}^2 , and find the local intersection number at their point of intersection.

Problem 3. Write T^k for the k-torus $\mathbb{R}^k/\mathbb{Z}^k$. On the space of 1-forms $\Omega^1(T^k)$, define an averaging operator $A: \omega \mapsto \int_{t \in T^k} L_t^*(\omega) d\mu$, where L_t means translation by tand $d\mu$ is Lebesgue measure. Prove that A defines an isomorphism from $H^1_{dR}(T^k)$ to the cotangent space $T_0^*(T^k)$.