Prelim, Numerical Analysis I, (fall course) 8/15/23, 10-12 AM

1. Assume the matrix A to be symmetric, strictly positive definite.

(a) Show that the matrix is nonsingular.
(b) Define its Cholesky decomposition and determine the number of multiplications needed in the solution.
(c) Give the pseudo inverse of a general matrix A and show how it can be used to solve over and under determined systems. Discuss properties of the solutions.

2. (a) Define Newton's method for minimization of a function $f(x), x \in R^d$, which has a unique minimum at x_0.

(b) Prove that the method converges if $d = 1$ and,

$$f(x) \in C^3, f''(x_0) \geq \varepsilon > 0, f'''(x) > 0 \; when \; x > x_0, f'''(x) < 0, x < x_0$$

(c) If the minimization is constrained such that $0 \leq x \leq 1$, describe a penalty method for the minimization of the constrained problem.

3. The midpoint rule for numerical integration is,

$$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right)$$

(a) Show that the midpoint rule is a Gauss quadrature method.

(b) Derive an error estimate based on the value of $f''\left(\frac{a+b}{2}\right)$.

(c) Describe Monte Carlo integration of the integral above with related error estimate.