Preliminary exam, Numerical Analysis, Part I, algebra and approximation
10:00 AM – 12:00 PM, January 7, 2023

1. (a) Prove that unitary transformations preserve the L2 norm and that similarity transformations preserve the eigenvalues of a matrix.
(b) Define Givens rotations and show how they can be used in the QR-method for finding the complete set of eigenvalues.
(c) Use the result from (a) to prove that the QR process preserves the set of eigenvalues and that the process is L2 stable.

2. Consider the constrained minimization problem,

$$\min_{(x_1,x_2)\in R^2,\ x_1\geq 0}((x_1+1)^2+(x_2-1)^2).$$

(a) Formulate Newton's method in detail for the unconstrained problem (without $x_1 \geq 0$).
(b) Show how Newton's method can still be used for the constrained problem if a penalty function is added.
(c) Use the Kuhn-Tucker theorem to compute the minimum and prove that this value satisfies the necessary Kuhn-Tucker conditions.

3. Consider numerical approximations of $\int_0^3 f(x)dx$.

(a) Derive an open 2-point Newton-Cotes quadrature formula by interpolation of $f(x)$ with a linear polynomial at the interpolation points, $x = 1$ and 2.
(b) What order in *h* is the above quadrature formula if $f(x)$ is sufficiently regular and the interval (0,3) is rescaled to (0,3*h*). Can a general 2-point formula with other interpolation points have higher order of accuracy?
(c) If the linear polynomial interpolation is replaced by Chebyshev polynomial interpolation with nodes at, $x = 0, 1.5, 3$, what would the quadrature formula be?