The University of Texas at Austin
Department of Mathematics

The Preliminary Examination in Probability
Part I

Thu, Jan 9, 2025

Problem 1 (30 pts)**.** Suppose that $n \geq 2$ points are chosen inside the unit interval $[0, 1]$, independently and with respect to the uniform distribution. Compute the moments $\mathbb{E}\big[L^k\big]$, $k \in \mathbb{N}$, of the length L of the smallest closed interval which contains all the points.

Problem 2 (40 pts)**.** Let $(S, \mathcal{S})$ be a measurable space, and let $\mathcal{P} = \mathcal{P}(S, \mathcal{S})$ denote the set of all probability measures on $(S, \mathcal{S})$. The **Hellinger distance** $H(\mu, \nu)$ between $\mu, \nu \in \mathcal{P}$ is defined by

$$H(\mu, \nu) = \left(\int \left(\sqrt{m} - \sqrt{n}\right)^2 d\rho \right)^{1/2} \text{ where } m = \frac{d\mu}{d\rho} \text{ and } n = \frac{d\nu}{d\rho},$$

and $\rho \in \mathcal{D}(\{\mu, \nu\})$, and where, for $\mathcal{R} \subseteq \mathcal{P}$, $\mathcal{D}(\mathcal{R})$ denotes the set of all $\rho \in \mathcal{P}$ such that $\nu \ll \rho$ for each $\nu \in \mathcal{R}$ (such measures are called **dominating measures** for $\mathcal{R}$).

(1) Prove that $\mathcal{D}(\mathcal{R}) \neq \emptyset$ if $\mathcal{R}$ is at most countable.
(2) Find an example of a measurable space $(S, \mathcal{S})$ and a subset $\mathcal{R}$ of $\mathcal{P}(S, \mathcal{S})$ such that $\mathcal{D}(\mathcal{R}) = \emptyset$.
(3) Show that and the value of $H(\nu, \mu)$ does not depend on the choice of $\rho \in \mathcal{D}(\{\mu, \nu\})$,
(4) Find the diameter $d = \sup\{H(\mu, \nu) : \mu, \nu \in \mathcal{P}\}$ of $\mathcal{P}$ and show that $H(\mu, \nu) = d$ if and only if μ and ν are mutually singular.

Problem 3 (30 pts)**.** Let X be a random variable with pdf $f_X(x) = e^{-x} 1_{\{x \geq 0\}}$, and let $X_1 = \min(X, 1)$. For $x \geq 0$, find the conditional distribution of X, given $X_1 = x$, and compute $\mathbb{E}[X \mid \sigma(X_1)]$.