The University of Texas at Austin
Department of Mathematics

The Preliminary Examination in Probability
Part I

Tue, Aug 20, 2024

Problem 1 (Growth of the expected maximum)**.** Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables bounded in $\mathbb{L}^p$, for some $p \geq 1$. Show that there exists a constant $C \geq 0$ such that

$$\mathbb{E}\Big[\max\Big(|X_1|, \dots, |X_n|\Big)\Big] \leq Cn^{1/p} \text{ for all } n \in \mathbb{N}.$$

(*Hint:* For each nondecreasing function f and all $a_1, \dots, a_n \in \mathbb{R}$ we have $f(\max(a_1, \dots, a_n)) \leq \sum_{k=1}^n f(a_k)$)

Problem 2 (Geometric random sums)**.** Let $\{X_k\}_{k\in\mathbb{N}}$ be an iid sequence of random variables, $\{p_n\}_{n\in\mathbb{N}}$ a sequence in $(0,1)$ with $\lim_n p_n = 0$, and $\{G_n\}_{n\in\mathbb{N}}$ a sequence of random variables, independent of $\{X_k\}_{k\in\mathbb{N}}$, such that

$$\mathbb{P}[G_n = N] = (1-p_n)^{N-1}p_n \text{ for } N \in \mathbb{N}.$$

(1) Derive an expression for the characteristic function of the random sum

$$S_n = p_n \sum_{k=1}^{G_n} X_k, n \in \mathbb{N},$$

in terms of the characteristic function φ of X_1.

(2) Assuming that $X_1 \in \mathbb{L}^1$ and $\theta = \mathbb{E}[X_1] > 0$, show that S_n converges in distribution as $n \to \infty$, and identify the limit.

Problem 3 (Conditioning Bernoullies)**.** Let X, Y be two Bernoulli random variables with parameter $p \in (0,1)$, i.e. $\mathbb{P}[X = 1] = \mathbb{P}[Y = 1] = 1 - \mathbb{P}[X = 0] = 1 - \mathbb{P}[Y = 0] = p$. Assuming that X and Y are independent of each other, compute the joint distribution of the random vector

$$\Big(\mathbb{E}[X \mid Z], \mathbb{E}[Y \mid Z], Z\Big) \text{ where } Z = 1_{\{X+Y=0\}}.$$