The University of Texas at Austin
Department of Mathematics

Preliminary Examination in Probability
Part II

August, 2024

Problem 2.1. Let B a standard, one-dimensional Brownian motion. Denote by

$$Z_t := e^{B_t - \frac{1}{2}t},\ T_b := \inf\{t > 0 |\ Z_t = b\} \text{ and } Z^* := \sup_{t \geq 0} Z_t.$$

Compute

(1) $\mathbb{P}[T_b < \infty]$ for $b > 1$,
(2) the law of Z^* and the law of $1/Z^*$. (**Hint:** compute the probability $\mathbb{P}[Z^* > b]$ for $b > 1$).

Problem 2.2. Let X be a continuous semi-martingale, and X^n a sequence of continuous processes of bounded variation such that, for each $t \geq 0$, we have

$$\mathbb{P}[\lim_{n\to\infty} X_t^n = X_t] = 1.$$

If $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1, show that

$$\lim_{n\to\infty} \int_0^t f(X_s^n) dX_s^n = \int_0^t f(X_s) dX_s + \frac{1}{2} \int_0^t f'(X_s) d\langle X \rangle_s,$$

holds $\mathbb{P}$-a.s. for every $t \geq 0$.

Problem 2.3. (Brownian bridge) Let $(B_t)_{0\leq t\leq 1}$ a standard one-dimensional Brownian motion (with time horizon $T = 1$) and denote by $(\mathcal{F}_t)_{0\leq t\leq T}$ the (augmented) filtration generated by B. Denote by

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma(B_1), \quad 0 \leq t \leq 1,$$

a new filtration (where the value of the BM at terminal time $T = 1$ is known at time t). Show that

(1)
$$\mathbb{E}[B_t - B_s | \mathcal{G}_s] = \frac{t-s}{1-s}(B_1 - B_s),\ 0 \leq s \leq t \leq 1.$$

(2) the process $(\beta_t)_{0\leq t\leq 1}$, defined by
$$\beta_t := B_t - \int_0^t \frac{B_1 - B_s}{1-s} ds,\ 0 \leq t \leq 1$$
is a $\mathcal{G}_t$-Brownian motion, independent of B_1.

(3) Denote by
$$X_t^x := xt + B_t - tB_1, \quad 0 \leq t \leq 1,$$
(the Brownian bridge ending at x at time $T = 1$).
Show that
$$X_t^x = \int_0^t \frac{x - X_s^x}{1-s} ds + \beta_t, \quad 0 \leq t \leq 1.$$

Note: one can obtain the representation of the Brownian bridge in (3) from (2) by conditioning the BM on its terminal value.