PRELIMINARY EXAMINATION:
ANALYSIS — Part II

January 11, 2024

Work all 4 of the following 4 problems.

1. Let f be a rational function with the property that $|f(z)| = 1$ whenever $z \in \mathbb{R}$. Let $\omega \in \mathbb{C}$. Show that f has a zero at ω if and only if $1/f$ has a zero at $\overline{\omega}$. [Hint: consider $g(z) = \overline{f(\overline{z})}$, and the product fg.]

2. Assume that $f : \mathbb{D} \to \mathbb{C}$ is holomorphic, with $Re(f(z)) > 0$ for all $z \in \mathbb{D}$. Moreover, assume that $f(0) = a > 0$. Prove that $|f'(0)| \leq 2a$. Does there exist such an f fulfilling equality, $|f'(0)| = 2a$?

Hint: Verify that the fractional linear transformation $T : z \mapsto \frac{z-a}{z+a}$ maps the right half plane to $\mathbb{D}$. Then, consider the function $g := T \circ f$.

3. Consider $z_n \in \mathbb{D} \subset \mathbb{C}$ such that z_n converges to 0 when n goes to infinity. Consider $\{f_n\}_{n \in \mathbb{N}}$ a sequence of holomorphic functions converging uniformly to f on $\mathbb{D}$ and such that z_n is the unique zero of f_n on $\mathbb{D}$. Show that either $f = 0$ for all $z \in \mathbb{D}$, or 0 is the unique zero of f.

4. Find all entire functions f that satisfy $f(\sqrt{n}) = n^2$ for every positive integer n, and $|f(z)| \leq e^{3|z|}$ for every complex number z.