PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part II

Saturday, January 13, 2024, 11:30am-1:30pm

Work all 3 of the following 3 problems.

1. For $d > 0$ an integer, consider the Sobolev space $H^s(\mathbb{R}^d)$. We assume that $s > d/2$.

(**a**) Define the usual norm for $H^s(\mathbb{R}^d)$.

(**b**) Show that $\displaystyle\int_{\mathbb{R}^d} (1 + |\xi|^2)^{-s}\, d\xi < \infty$.

(**c**) Use (b) to prove that $H^s(\mathbb{R}^d)$ is continuously embedded in $L^\infty(\mathbb{R}^d)$. [Hint: For $f \in H^2(\mathbb{R}^d)$, first write f as the Fourier inversion integral of $\hat{f}$.]

(**d**) Use (c) to show that every $f \in H^s(\mathbb{R}^d)$ is almost everywhere equal to a continuous function. [Hint: The Schwartz space is dense in $H^s(\mathbb{R}^d)$.]

2. Let $\Omega \subset \mathbb{R}^2$ be a domain with a smooth boundary and consider the variational problem: Find $u \in V$ such that

$$(u, v) + (\nabla \cdot u, \nabla \cdot v) = (f, \nabla \cdot v) \quad \text{for all } v \in V,$$

where u and v are vectors in $\mathbb{R}^2$,

$$(u, v) = \int_\Omega \big(u_1(x)\, v_1(x) + u_2(x)\, v_2(x)\big)\, dx, \quad \text{and} \quad \nabla \cdot u = \operatorname{div} u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}.$$

(**a**) For the problem to make sense, define V and a space for f. Why is V a (real) Hilbert space?

(**b**) State the Lax-Milgram theorem for Hilbert spaces.

(**c**) Show that the hypotheses of the Lax-Milgram theorem hold for this problem. What norm do we use for V?

3. Let X and Y be Banach spaces, and let F and G take X to Y be C^1.

(**a**) Let $H(x, \epsilon) = F(x) + \epsilon G(x)$ for $\epsilon \in \mathbb{R}$. If $H(x_0, 0) = 0$ and $DF(x_0)$ is invertible, show that there exists $x \in X$ such that $H(x, \epsilon) = 0$ for ϵ sufficiently close to 0. [Hint: apply the Implicit Function Theorem.]

(**b**) For small ϵ, prove that there is a solution $w \in H^2(0, \pi)$ to

$$w'' = w + \epsilon w^2, \quad w(0) = w(\pi) = 0.$$