Applied Mathematics Preliminary Exam, Part A
January 13, 2024

Work 4 of the following 5 problems.

1. Consider $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ measurable. Let $p, q \in (1, \infty)$ with $p^{-1} + q^{-1} = 1$. Given $K \in \mathrm{L}^q(U \times V)$, define

$$(Af)(u) = \int_V K(u,v) f(v)\, dv\,, \qquad f \in \mathrm{L}^p(V)\,, \quad u \in U\,.$$

Show that this defines a continuous linear operator $A : \mathrm{L}^p(V) \to \mathrm{L}^q(U)$, that the operator norm of A satisfies $\|A\| \le \|K\|_{\mathrm{L}^q}$, and that A is in fact compact.

2. Prove that a closed linear subspace of a reflexive Banach space is reflexive as well.

3. Let X be a Banach space. Let $A : X \to X$ and $B : X' \to X'$ be linear. Assume that $(B\varphi)(x) = \varphi(Ax)$, for all $x \in X$ and all $\varphi \in X'$. Show that A and B are continuous.

4. Let A be a continuous linear operator from a Banach space X to a normed vector space Y. Prove that A is invertible if and only if both A and $A' : Y' \to X'$ are bounded below.

5. Let T be a distribution in $\mathcal{D}'(\mathbb{R})$ whose support is $\{0\}$, meaning that $Tf = 0$ for every function $f \in \mathcal{D}(\mathbb{R})$ whose support does not contain 0. Show that there exists $n \ge 0$, such that $Tg = 0$ whenever $g \in \mathcal{D}(\mathbb{R})$ satisfies $g(0) = g'(0) = \ldots = g^{(n)}(0) = 0$.

Hint. Consider gh_ε in place of g, where $h_\varepsilon(x) = h(x/\varepsilon)$ and $h \in \mathcal{D}(\mathbb{R})$ is identically 1 near the origin.