PRELIMINARY EXAMINATION IN ANALYSIS
PART I
JANUARY 2024

Please provide complete proofs for at least 3 of the following 4 problems.

(1) Let (X, d) be a compact metric space and let μ be a Borel probability measure on X. Let $f : X \to \mathbb{R}$ be a Borel measurable function. Prove that for every $\epsilon > 0$ there exist a closed subset $Y \subset X$ such that $\mu(Y) > 1 - \delta$ and the restriction of f to Y is continuous. Do not use Lusin's Theorem. The point of this exercise is to prove this from ideas which one typically learns prior to Lusin's Theorem.

(2) Suppose ν and μ are positive finite measures on $(X, \mathcal{M})$ (where $\mathcal{M}$ is a sigma-algebra on X) with ν absolutely continuous to μ (in the sense that for every $E \in \mathcal{M}$ with $\mu(E) = 0$, we have $\nu(E) = 0$). Prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that if $E \in \mathcal{M}$ and $\mu(E) \leq \delta$ then $\nu(E) \leq \epsilon$.

(3) Let H be a Hilbert space. Given a linear subspace $S \subset H$, we define $S^{\perp} = \{x \in H : \langle x, y \rangle = 0 \ \forall y \in S\}$. Let $S^{\perp\perp} = (S^{\perp})^{\perp}$. Prove $S^{\perp\perp}$ is the closure of S in the norm topology.

(4) Let (X, d) be a compact metric space. Let $\mathrm{Prob}(X)$ be the set of Borel probability measures on X, $C(X)$ be the Banach space of continuous functions on X and $C(X)^*$ be its Banach dual. Given $\mu \in \mathrm{Prob}(X)$, let $\phi_\mu : C(X) \to \mathbb{R}$ be the functional $\phi_\mu(f) = \int f \, d\mu$. Prove that the set $K = \{\phi_\mu : \mu \in \mathrm{Prob}(X)\} \subset C(X)^*$ is compact in the weak* topology.