ALGEBRA PRELIMINARY EXAM: PART II

JANUARY 14TH, 2022

Solve at least three of the following five problems.

Problem 1.

(a) Let $\mathbb{F}_{3^n}$ denote the field with 3^n elements. Find the smallest n such that $\mathbb{F}_{3^n}$ contains a primitive 11th root root of unity, i.e, such that there is an element $x \in \mathbb{F}_{3^n}$ with $x^{11} = 1$ and $x \neq 1$.

(b) For integers n and m, write $(n + m\sqrt{3})^{24}$ as $a + b\sqrt{3}$ for $a, b \in \mathbb{Z}$. Show that b is divisible by 7.

Problem 2.

(a) Let K/k be a Galois extension of fields. Show that $x \in K$ is a primitive element if and only if its stabilizer in the Galois group is trivial.

(b) Let $k = \mathbb{Q}$ and let K be the splitting field of the polynomial $t^3 - 2$. Find a primitive element for this field extension.

Problem 3.

(a) Suppose $f(t) \in \mathbb{Q}[t]$ is an irreducible polynomial of degree 5 with exactly 3 real roots. Show that the Galois group of f is the full symmetric group S_5.

(b) Show that the equation $t^5 - 10t - 5 = 0$ is not solvable by radicals.

Problem 4. For every integer $n \geqslant 1$, show that there is a Galois extension $K/\mathbb{Q}$ with $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/n$.

Problem 5.

(a) Let $f(t) \in \mathbb{F}_p[t]$ be an irreducible polynomial of degree d. Find its Galois group.

(b) Show that $f(t) = t^p - t - 1 \in \mathbb{F}_p[t]$ is irreducible.