Real Analysis Prelim Spring 2022

January 2022

Problem 1. Let $\mathbb{Z}$ be a subset of $\mathbb{R}$ with measure zero.
Show that the set $A = \{ x^2 \ : \ x \in \mathbb{Z}\}$ also has measure zero.

Problem 2. Let $f : \mathbb{R}^n \to [0, +\infty]$ a measurable function, denote the measure of set $\Omega \in \mathbb{R}^n$ by $|\Omega|$. Show that

a) $| \{x \in \mathbb{R}^n \ : \ f(x) \geq k\} | \ \leq \frac{1}{k} \int f.$

b) If f is integrable, then $|\{x \in \mathbb{R}^n \ : \ f(x) = +\infty\} | = 0.$

Problem 3. Let f be of bounded variation on an interval $[a, b]$. If $f = g + h$, where g is absolutely continuous and h is singular, show that

$$\int_a^b \phi \, df \ = \ \int_a^b \phi \, f' dx \ + \ \int_a^b \phi \, dh \,, \tag{1}$$

for any continuous ϕ.

Problem 4. Let $\{f_k\}$ be a sequence of non negative measurable function defined on the measurable set $E \in \mathbb{R}^n$.
If $f_k \to f$ and $f_k \leq f$ a.e., show that $\int_E f_k \ \to \ \int_E f$.

Problem 5. Let $1 \leq p, q \ \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g$ is bounded and continuous function on $\mathbb{R}^n$.