PRELIMINARY EXAMINATION IN ANALYSIS

Part II, Complex Analysis

January 11, 2022

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1. Let K be a compact connected set in $\mathbb{C}$ that contains the points $\pm i$. Show that there exists a single-valued analytic branch of $(z^2+1)^{-1/2}$ on $\mathbb{C} \setminus K$ and determine the possible values of its integral along a closed regular curve in $\mathbb{C} \setminus K$.

2. Let f be analytic and bounded in the upper half plane $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Assume that $f(z+1) = f(z)$ for all $z \in H$. Prove that $f(z)$ has a limit as $\operatorname{Im} z \to +\infty$.

3. Let $0 < \alpha < 1$. Show that $\prod_{n=0}^{\infty} \cos(\alpha^n z)$ defines an entire function f of finite order. Determine the order and genus of f.

4. Let U and V be two disjoint non-empty open subsets of $\mathbb{C}$. Let $n \mapsto f_n$ be a sequence of analytic functions $f_n : U \to V$. Show that some subsequence converges, locally uniformly, to an analytic function f, and that f is one-to-one if each f_n is one-to-one.

$\big($*Remark added after the exam.* An extra condition is missing: the sequence $n \mapsto f_n(u)$ should be bounded for some $u \in U$.$\big)$