PRELIMINARY EXAMINATION: APPLIED MATHEMATICS — Part I

January 10, 2022

Work all 3 of the following 3 problems.

1. Let X be an NLS and $\{x_n\}_{n=1}^{\infty}$ be a sequence from X.

(**a**) If $x_n \to x$ as $n \to \infty$, prove that $x_n \rightharpoonup x$.

(**b**) If $\{x_n\}_{n=1}^{\infty}$ converges weakly as $n \to \infty$, prove that its weak limit is unique.

(**c**) If $x_n \overset{w}{\rightharpoonup} x$ as $n \to \infty$, prove that $\{\|x_n\|_X\}_{n=1}^{\infty}$ is bounded. [Hint: use the Uniform Boundedness Principle.]

(**d**) If $x_n \overset{w}{\rightharpoonup} x$, prove that $\|x\| \le \liminf_{n\to\infty} \|x_n\|$. [Hint: use one of the corollaries of the Hahn-Banach Theorem.]

2. For a vector space V, recall that $B \subset V$ is a *Hamel basis* if every element of V can be expressed uniquely as a finite linear combination of the vectors in B.

(**a**) State the Baire Category Theorem.

(**b**) Prove that an infinite dimensional Banach space X cannot have a countably infinite Hamel basis. [Hint: suppose $\{e_n\}_{n=1}^{\infty}$ is a Hamel basis and consider $X_n = \text{span}\{e_1, \ldots, e_n\}$ (show that X_n has empty interior).]

3. Spectral theory.

(**a**) Suppose X and Y are Banach spaces and $T \in B(X, Y)$ is bounded below. Prove that T is one-to-one and $R(T)$ is closed in Y.

(**b**) Let H be a Hilbert space and $T \in B(H, H)$ be a self-adjoint operator. Prove that $\langle Tx, x\rangle \in \mathbb{R}$ for all $x \in H$.

(**c**) Let H be a Hilbert space and $T \in B(H, H)$ be a self-adjoint operator. Prove that the spectrum $\sigma(T) \subset \mathbb{R}$.