The University of Texas at Austin
Department of Mathematics

The Preliminary Examination in Probability
Part I

Thu, Jan 13, 2021

Problem 1. Suppose that $\{X_n, n \geq 1\}$ is a sequence of i.i.d nonnegative random variables, that is, $X_n \geq 0$ a.s. If $\mathbb{E}(X_1) = \infty$, show that $\frac{1}{n}\sum_{k=1}^{n} X_k \to \infty$ a.s.

Problem 2. Let μ be a probability measure on $\mathbb{R}$ and let φ be its characteristic function. Show that μ has no atoms if

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-ita}\varphi(t)\,dt = 0 \text{ for all } a \in \mathbb{R}.$$

Problem 3. Let $X \in \mathbb{R}$, $\mathbb{E}(|X|) < \infty$. Let $\mathcal{F}$ be a sub-σ-algebra. Show that there exists a sequence $\{\mathcal{P}_n : n \geq 1\}$ of finite partitions of Ω such that $\sigma(\mathcal{P}_n) \subseteq \sigma(\mathcal{P}_{n+1}) \subseteq \mathcal{F}$ for all $n \geq 1$, and

$$\lim_{n\to\infty} \mathbb{E}\left(\left|\sum_{A\in\mathcal{P}_n} \mathbb{E}(X|A)\mathbf{1}_A - \mathbb{E}(X|\mathcal{F})\right|\right) = 0.$$