ALGEBRA I QUALIFYING EXAM

AUGUST 17, 2022

A passing score is 20/30.

Problem 1.

(a) Let A be a Euclidean domain with Euclidean function $\delta : A \to \mathbb{Z}^{\geqslant 0}$. Show that A is a PID. (2.5 points.)

(b) Let A be a PID. Show that A is a UFD. (5 points.)

(c) Let $d \in \mathbb{Z}$ be an integer congruent to 1 modulo 4. Let $A = \mathbb{Z}[\sqrt{d}]$ and let F denote the field of fractions of A.

Show that the polynomial $t^2 - t + \frac{1-d}{4}$ is irreducible in $A[t]$ but is not irreducible in $F[t]$. Deduce that A is not a UFD. (2.5 points.)

Problem 2.

(a) Let G be a finite group and let $H \trianglelefteq G$ be a normal subgroup.

Let $P \subseteq H$ be a p-Sylow subgroup of H. Let $N_G(P)$ (resp. $N_H(P)$) denote the normalizer of P in G (resp. in H).

Show that the natural map $H/N_H(P) \to G/N_G(P)$ is a bijection. Deduce that $H \cdot N_G(P) = G$. (5 points.)

(b) Let K be a finite group and let $P \subseteq K$ be a p-Sylow subgroup. Show that $N_K(N_K(P)) = N_K(P)$. (Hint: use (a).) (5 points.)

Problem 3. Let G be a group of order $833 = 7^2 \cdot 17$. Let S be a set of order 32 on which G acts.

(a) Show that there necessarily exists a fixed point for the action, i.e., there exists $x \in S$ with $g \cdot x = x$ for all $g \in G$. (5 points.)

(b) For given G of order 833, show that there necessarily exists a set S with a G-action such that $|S| = 32$ and there exists *exactly* one fixed point. (5 points.)