ALGEBRA PRELIMINARY EXAM: PART II

A passing score is 20/26.

Problem 1

Let p and q be two primes. Set $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$.
Compute the number of irreducible polynomials in $\mathbb{F}_p[x]$ of degree q. (5 points)

Hint: Use the classification of extensions of $\mathbb{F}_p$.

Problem 2

Let $\alpha = \sqrt{2} + \sqrt{3}$.

a) Prove that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois and determine its Galois group as an abstract group. (4 points)

b) Determine the minimal polynomial $f(x)$ of α over $\mathbb{Q}$. (2 points)

c) Prove that $f(x)$ is reducible modulo p for every prime p. (3 points)

d) Prove that $\sqrt[3]{2} \notin \mathbb{Q}(\alpha)$. (2 points)

Problem 3

Let $n \in \mathbb{N}$ and ζ_n a primitive n-th root of unity.

a) Viewing complex conjugation τ as an element of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, prove that $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ is the fixed field of subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ generated by τ. (3 points)

b) Find a polynomial $f(x) \in \mathbb{Q}[x]$ whose Galois group over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}/5\mathbb{Z}$. It is sufficient to identify the polynomial in $\mathbb{C}[x]$ and argue that its coefficients are rational. (3 points)

c) Prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of $\mathbb{Q}(\zeta_n)$.(4 points)

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