Real Analysis Prelim Fall 2022 - August 11, 2022

Choose any three out of the following four problems:

1. Let $0 < q < p < \infty$. Let $E \subset \mathbb{R}^n$ be measurable with measure $|E| < \infty$. Let f be a measurable function on $\mathbb{R}^n$ such that $N \overset{\text{def}}{=} \sup_{\lambda>0} \lambda^p |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|$ is finite.

(a) Prove that $\int_E |f|^q$ is finite.

(b) Refine the argument of (a) to prove that

$$\int_E |f|^q \le C N^{q/p} |E|^{1-q/p}$$

where C is a constant that depends only on n, p, and q.

2. Let $p \in [1, \infty)$ and suppose $\{f_n\}_{n=1}^{\infty} \subset L^p(\mathbb{R})$ is a sequence that converges to 0 in the L^p norm.

Prove that one can find a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to 0$ almost everywhere.

3. For a function $f \in \mathrm{L}^1(\mathbb{R}^2)$ let $\widetilde{M}f$ be the unrestricted maximal function

$$\widetilde{M}f(x_0, y_0) = \frac{1}{|Q|} \sup_Q \int_Q |f(x,y)| dxdy,$$

where the supremum is over all $Q = [x_0 - k, x_0 + k] \times [y_0 - l, y_0 + l]$ with $k, l > 0$.

(a) Show that $\widetilde{M}f(x_0, y_0) \le M_1 M_2 f(x_0, y_0)$, where

$$M_1 f(x_0, y) = \sup_{k>0} \frac{1}{2k} \int_{x_0-k}^{x_0+k} |f(x,y)| dx, \quad M_2 f(x, y_0) = \sup_{l>0} \frac{1}{2l} \int_{y_0-l}^{y_0+l} |f(x,y)| dy.$$

(b) Show that there exists $C > 0$ such that if $f \in \mathrm{L}^2(\mathbb{R}^2)$, then

$$\|\widetilde{M}f\|_{\mathrm{L}^2(\mathbb{R}^2)} \le C \|f\|_{\mathrm{L}^2(\mathbb{R}^2)}$$

4. Let f and the sequence $\{f_k\}_{k\ge1}$, be in L^p, for $1 \le p < \infty$.

If $f_k \to f$ pointwise a.e. and $\|f_k\|_p \to \|f\|_p$, show that $\|f - f_k\|_p \to 0$.