PRELIMINARY EXAMINATION:
APPLIED MATHEMATICS — Part I

August 10, 2022

Work all 3 of the following 3 problems.

1. Let X be a normed linear space and X^* its dual space. Let $\{f_n\}_{n=1}^{\infty} \subset X^*$ and $f \in X^*$.

(**a**) Define what it means for f_n to converge weak-* to f.

(**b**) Prove that if f_n converges converges weak-* to f, then f is unique.

(**c**) State the Uniform Boundedness Principle.

(**d**) Suppose that X is a Banach space. Prove that $\{\|f_n\|_{X^*}\}_{n=1}^{\infty}$ is bounded.

2. Let X be a normed linear space and Y a finite dimensional subspace of X. For $x \in X$ and $S \subset X$, let

$$d(x, S) = \inf_{z \in S} \|x - z\|$$

denote the distance from x to S. Fix $x_0 \in X$ and let $\mathcal{B} = \{y \in Y : \|y\| \leq 3\|x_0\|\}$.

(**a**) Show that

$$d(x_0, Y) = d(x_0, \mathcal{B}).$$

[Hint: first show that $\|x_0\| \geq d(x_0, \mathcal{B}) \geq d(x_0, Y)$.]

(**b**) Show that there is some $y_0 \in \mathcal{B} \subset Y$ such that

$$d(x_0, y_0) = d(x_0, Y).$$

We say that y_0 is a *best approximation* to x_0. [Hint: why is $\mathcal{B}$ compact?]

(**c**) Show by example that a best approximation may not be unique. [Hint: try $X = (\mathbb{R}^2, \|\cdot\|_{\ell^1})$ and $Y = \text{span}(1, 1)$.]

3. Let X be a Banach space, $S, T \in B(X, X)$, and I be the identity map on X. Suppose further that T is compact.

(**a**) Prove that TS and ST are compact.

(**b**) Describe the spectrum of a compact operator.

(**c**) If S is invertible and $S + T$ is injective, show that $S + T$ is invertible on all of X.